Age Dynamics and Optimal Recruitment Policies of Learned Societies. An Application to the Austrian Academy of Sciences*

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Abstract

In a hierarchical, constant sized organization the annual intake is strictly determined by the number of deaths and a statutory retirement age. Faced with a rising life expectancy there is a fundamental dilemma of two conflicting goals of such populations, e. g. an Academy of Sciences: to keep a young age structure while guaranteeing a high recruitment rate. In this paper we reconstruct the population of the Austrian Academy of Sciences from 1847 to 2005. Based on alternative scenarios of the age distribution of incoming members we then project the population of the Austrian Academy forward in

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time. We introduce an age-structured optimal control model to determine the optimal trade-off between the above mentioned conflicting goals. Our results indicate that it is optimal to elect either young or old aged new members. Moreover, we discuss some interesting policy implications of the obtained optimal recruitment policy (scientific excellence and life-long achievements).

1 Introduction

The dynamics of small populations are a neglected field in mathematical demography, despite its importance in several branches of science. The development of learned societies, universities, armies or other hierarchical organizations can be studied by methods of population dynamics and intertemporal optimization. The purpose of this paper is to show how methods of demography can be applied to reconstruct and project learned societies, in particular the Austrian Academy of Sciences. Moreover we also demonstrate the use of age structured optimal control theory to derive optimal recruitment policies. Since age is one central variable in population dynamics (as seniority is in manpower planning) the appropriate tool of controlling such systems is distributed parameter control (see, e.g., Feichtinger and Hartl 1986, Appendix A.5).

A fundamental dilemma in population dynamics is that a closed population can either grow or become older. Thus, ageing is the logical "fate" of a constant sized population (if premature death is excluded). This is not only true for "large" population, but also holds for subpopulations.

The increasing ageing of learned societies is a problem in many national academies. Faced with rising life expectancy, particularly for older persons, the average age of academy members increases. Another reason for the rightwards shift of the age distribution is an increase in the age at election (Leridon 2004).

The starting point of our paper was formed by a number of studies on the "Académie des Sciences (Institut de France)" (Leridon 2004), the "Royal Danish Academy of Sciences and Letters" (Matthiessen 1998) and the "National Academy of Sciences" [of the US] (Cohen 2003). These studies also examined the subject of learned societies by analysing their historic, present and future demographic development - in particular in the context of increasing ageing and declining mortality.

The population dynamics in hierarchical groups, whose total membership size remains constant, is determined by the rate of intake, the age distribution at entry into a given status, the number of exits (deaths or dismissals), a statutory retirement age, and the change in total population size. The intake itself is strictly determined by the number of deaths and a statutory retirement age[1]. The only degree of freedom one has is the age of entries, e.g. at election. To counteract the trend of ageing new members have to be elected at increasingly younger ages. However, this would have the drawback of reducing the rate of membership renewal. There is a fundamental dilemma of two conflicting goals of a constant-sized age structured population, e.g. an academy of sciences: to keep a young age-structure while to guarantee a high recruitment rate.

This trade-off may be illustrated by the following thought experiment - neglecting mortality for the moment. If an academy with an statutory retirement age, say at age 70, is established by electing only members at age 40, there will be no renewal until the members reach the retirement age 70. On the contrary, if all members are elected at age 69, then there will be maximum recruitment every year.

After describing the development of the age dynamics of the Austrian Academy of Sciences (AAS) we present several projections for the future age-structured development of the AAS. Based on four alternative scenarios for the age distribution of incoming members we project the AAS forward in time to study the sensitivity of the total number of members, their age distribution and the number of recruits for those alternative scenarios. Next, we introduce an age-structured optimal control model to determine the optimal trade-off between the rate of replacement and the mean age of a constant-sized population, whose dynamics is modelled by the McKendrick partial differential equation. A variant of Pontryagin's maximum principle is derived and used to determine the optimal recruitment distribution in the stationary case. It turns out that due to an U-shaped age-specific shadow price of a member it is optimal to elect either young or old aged new members. Finally, we discuss some interesting policy implications of the obtained optimal recruitment policy (scientific excellence and life long achievements).

We conclude our paper by stressing the fact that the proposed solution method may be used in other parts of population research as well. Among them are immigration with below-replacement fertility, manpower planning, university management, and control of the HIV/AIDS epidemic. Dynamic optimization techniques for age and duration - structured populations might play an increasing role in demography,

[1]Such an age limit has been introduced into the byelaws of the Austrian Academy of Sciences: From 1950 onwards members, who exceed the age of 75 years, did not count for the computation of the maximum number of members any longer, whereas they still kept their full rights. This threshold age was lowered to 70 years in 1972. Several academies cope with their "over-ageing" by a fixed age of exit that frees seats.

particularly within the theory of population policy. Decision support methods will be an interesting branch of future demography.

2 Demographic Evolution of the Austrian Academy of Sciences 1847-2005

Reconstructing the data

The data come from the biographic records from the members of the Austrian Academy of Sciences (Hittmair and Hunger 1997). The Austrian Academy of Sciences was founded in 1847 as the 'Kaiserliche Akademie der Wissenschaften in Wien' under the auspices of emperor Ferdinand I. The Academy is structured around two sections - the Section for Mathematics and the Natural Sciences (mat. nat.) and the Section for the Humanities and the Social Sciences (hum. soc.). Membership distinguishes between honorary members, full members and corresponding members. The latter category further distinguishes between corresponding members residing in Austria and corresponding members abroad. Full membership requires residence in Austria. If a full member moves abroad, his or her status changes to that of a corresponding member abroad while living outside of Austria.

The biographic records include date of birth and death, year of election, section membership, membership status and year of change of membership status. Exits from the Academy were mainly through death with few exceptions. Four full members, two corresponding members, and one corresponding member abroad voluntarily resigned from the Academy. During the Nazi period, several Jewish members were excluded from the Academy: between autumn 1938 and autumn 1940, six full members and 15 corresponding members (including corresponding members abroad) were expelled on 'racial' grounds (Matis 1997). After the end of World War II, the excluded members were listed again in the membership directories. Nowadays membership directories do not state those exclusions. When reconstructing the member populations during that time, we decided to keep these excluded members in the member population because we do not want to replicate the injustice committed at the time.

In 1945 those members of the Academy who had been members of the NSDAP were suspended. With the amnesty law in 1948, however, the membership of almost all of them was restored again. Nowadays membership directories do not give any information about these suspensions either.[2]

For the purposes of reconstructing the membership population, we assumed that elections always took place in mid-May each year, which has been the case at least in recent years. If members moved abroad or returned to Austria, we coded the change in membership status to take place in the middle of the year due to lack of information on the actual month of change.

Thereby, it is possible to reconstruct in particular those changes of the academy's statutes chronologically, which led to new class sizes, age limits and rules of election. Table 1 summarizes (the amendments of) the byelaws and gives the year, in which the respective change became effective. Apart from the increase of the maximum number of members in 1848, 1925 and 1992 threshold ages were introduced. From 1950 onwards members, who were older than 75 years, did not count for the computation of the maximum numbers any longer, whereas they still kept their full rights. This threshold age was lowered to 70 years in 1972.

In the following we will examine the population of the 'Austrian Academy of Sciences' in more detail, whereby we particularly focus on the full members. Since these are equipped with the right to vote, they may influence the current member structure by electing new members of a certain age group.

Population- and age structure over time

Figure 1 shows the development of full members from 1847 to 2005. In addition, those years are marked, where the amendments of the byelaws took place. After a quite stable process in the first half of the academy's existence, the amendments of the byelaws led to a continuous rise in the numbers of members. Besides the direct increases of the maximum number of members for each section in the years 1848, 1925 and 1992, the introduction of the threshold age (1950) as well as its reduction (1972) created a considerable space for potential members to be elected. This led to batch-wise growth of the population. Consequently, the academy started with 39 full members in 1847 and showed 167 full members in 2005. From the latter 88 full members belong to the mat.nat. section, whereas 79 are members of the hum.soc. section.

A detailed picture of the age development over time is plotted in Figure 2, which contains the average age of full academy members with and without consideration

[2] A thorough description of the activities of the Austrian Academy of Sciences from 1938–1945 can be found in Matis (1997).

Table 1: Changes in the byelaws concerning the number of members per class, age limits and rules of elections and the year they became operative.

Byelaws	**Year**[a]	**Members per section**	**Specific rules**
Statutes, 1st version (1847)	1847	24	24 members per section
- Addendum (1848)	1848	30	increase by 6 members per section
Byelaws, 1st version (1922)			
- Addendum (1925)	1925	33	increase by 3 members per section
Byelaws, 3rd version (1946)			
- Addendum (1949)	1950	33	age limit of 75 years; maximum of five new members elected per year
- Addendum (1960)	1961	33	restriction on elections dropped
- Addendum (1971)	1972	33	age limit of 70 years
- Addendum (1991)	1992	45[b]	increase by 12 members per section

[a] Year of election, where the change became effective for the first time.
[b] Informal agreement to distribute additional elections over three years.
(Schlögl 1992)

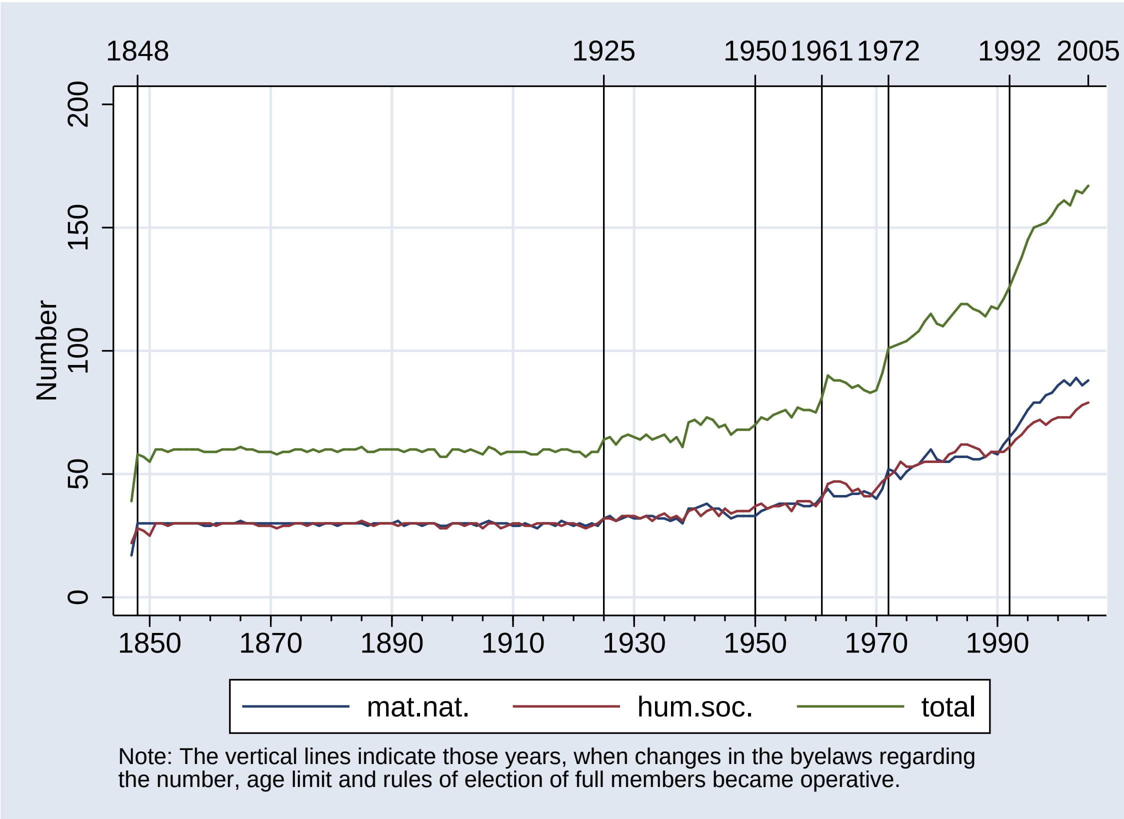

Figure 1: Number of full members by section, 1847-2005 (at midyear).

of the introduction of the threshold age of 75 years (1950) and its reduction towards 70 years (1972). Starting from 50 (mat.nat. section) and 52 (hum.soc. section) years in 1849 the average age rose to 66 (mat.nat. section) and 68 (hum.soc. section) years during a time span of one hundred years. With the introduction of the threshold age the development splits. Thus, while the overall average age continues at a higher level compared to the average age of those full members below the threshold age, the latter experiences an additional impulse of rejuvenation due to the decreased age limit. In 2005 all members of the mat.nat. (hum.soc.) section are 71 (70) years old on average.

Focussing on broader age groups shows, that the age group of the 60 to 69-year olds dominates, while that of the over-80-year olds steadily increases, which is due to rising life expectancy and an increasing age at election (figure not shown).

The duration of membership as a full member is determined by the entrance age on the one hand as well as by the time of death on the other hand. In spite of partly

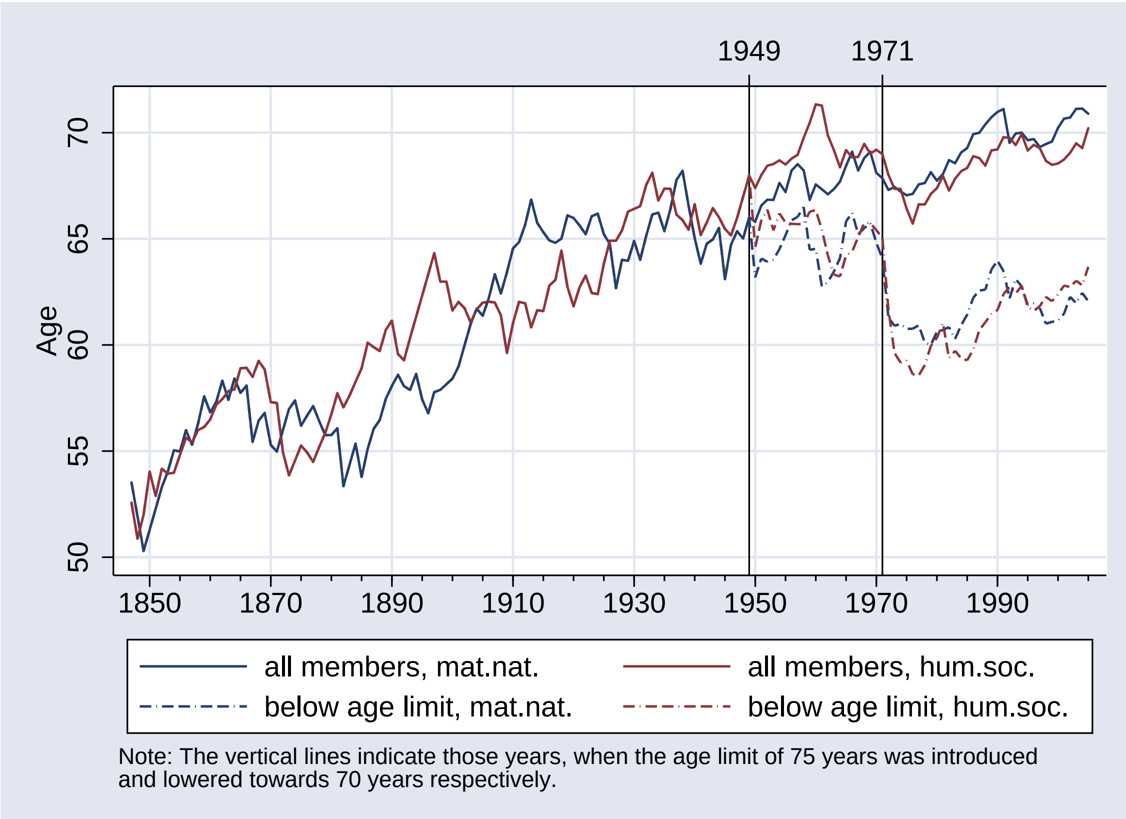

Figure 2: Mean age of full members per section over time (at midyear).

strong variations over time - which are caused by the small sample size - the mean period spent in the population by full members fluctuates around a value of 20 years during the first half of the 20th century. Afterwards, it rose from 16 years in 1946-50 to the average of 27 years for all members, while the mean tenure of full members below the threshold age lasts 12 years at the end (figure not shown).

The current age structure of the academy population is determined by its 'internal' ageing process as well as by the age structure of 'new entrants' (through elections) and the pattern of 'retirements' (constituted by deaths). In the following two sections we briefly review the dynamics of new entrants and deaths.

New Entrants

The number of elected members per section since the establishment of the academy is represented in Figure 3. While the number of elected members for each 5-year period up to the beginning of the 20th century amounted to 10-20 members, this

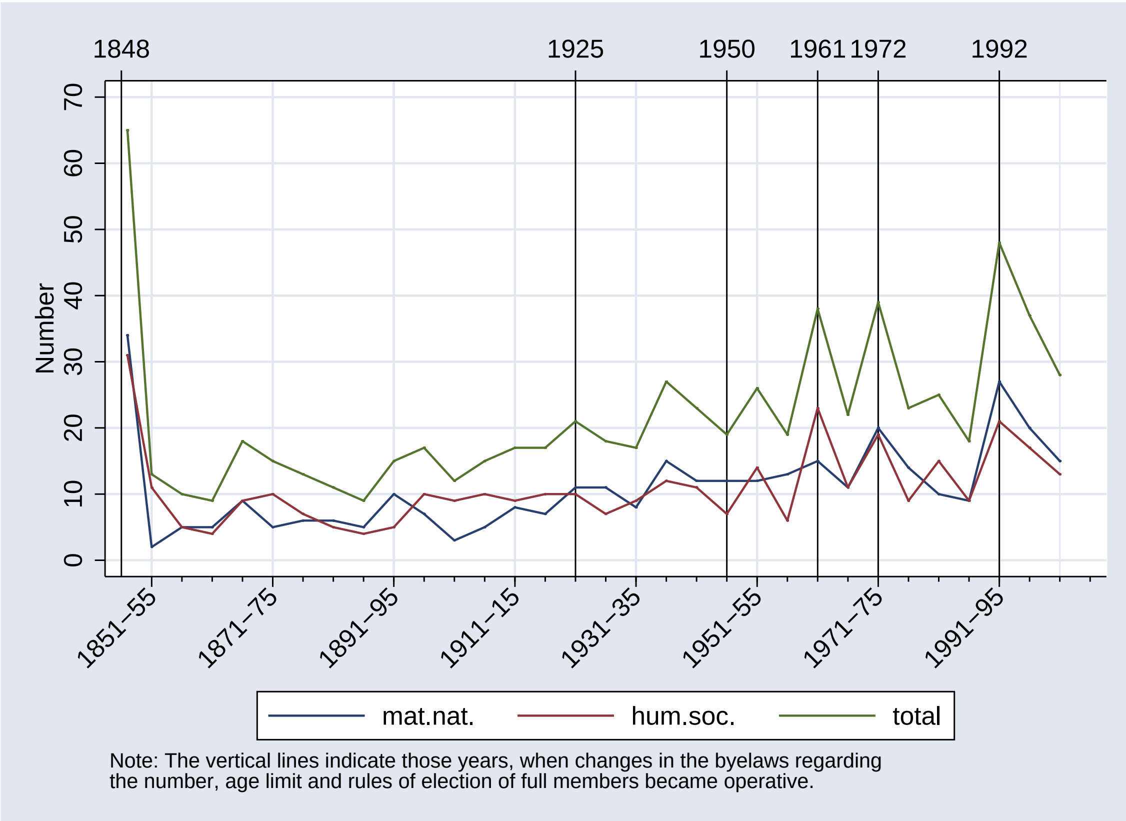

Figure 3: Number of elected members per section (5-year periods).

number even rose to over 20-25 later on. Clear peaks show up in particular during the amendments of the byelaws around 1961, 1972 and 1992, as it became possible to select a larger number of new members at one single date. Hence, 23 new full members were elected during the years 1961-62, 13 new full members in 1972 and 43 new full members from 1992 to 1995.

The illustration of the average age at the time of election is depicted in Figure 4. After its lowest value of about 40 years in 1861-65, the average age of elected full members rose continuously and reached more than 60 years several times in the second half of the 20th century. In the recent past it decreased again due to the reduction in the threshold age that led to the election of younger members.

The low mortality of academy members

In accordance to the generally observed population ageing, Figure 5 clarifies, that the average age at death - which determines the end of membership in the academy -

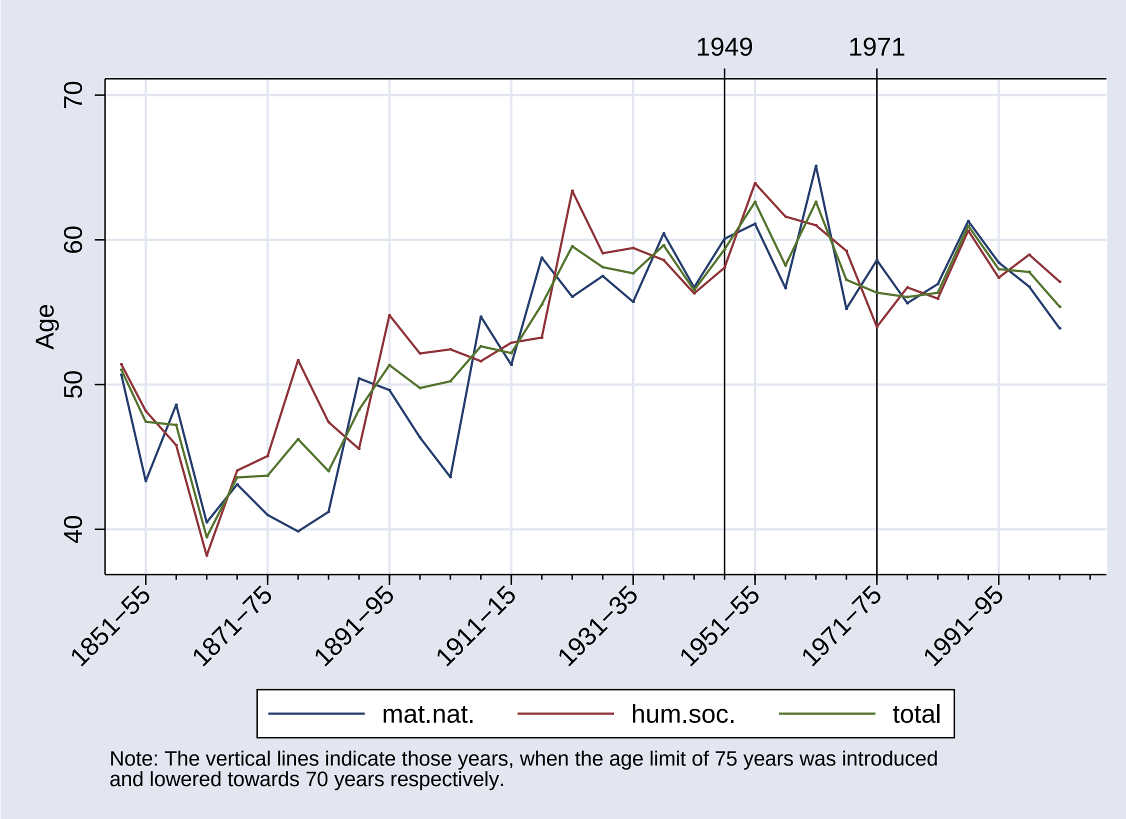

Figure 4: Mean age of full members at time of election (5-year periods).

rose continuously within our subpopulation. In 1851-55 a full academy member died at the average age of 63 years, while the age at death was 74 years in 1931-35 and 82 years in 1996-2000. As a consequence the increasing mean age at death contributed to the rising average period spent as a full member.

3 Projections of the future structure and size of the Academy

Over-ageing of professional organizations or bodies has been frequently seen as a disadvantage. An Academy of Sciences as an advisory body should stay in touch with the community of working researchers. During the last decades several new important branches of sciences, e.g. in informatics, biology, ecology, etc., developed which should be represented by young dynamic scientists.

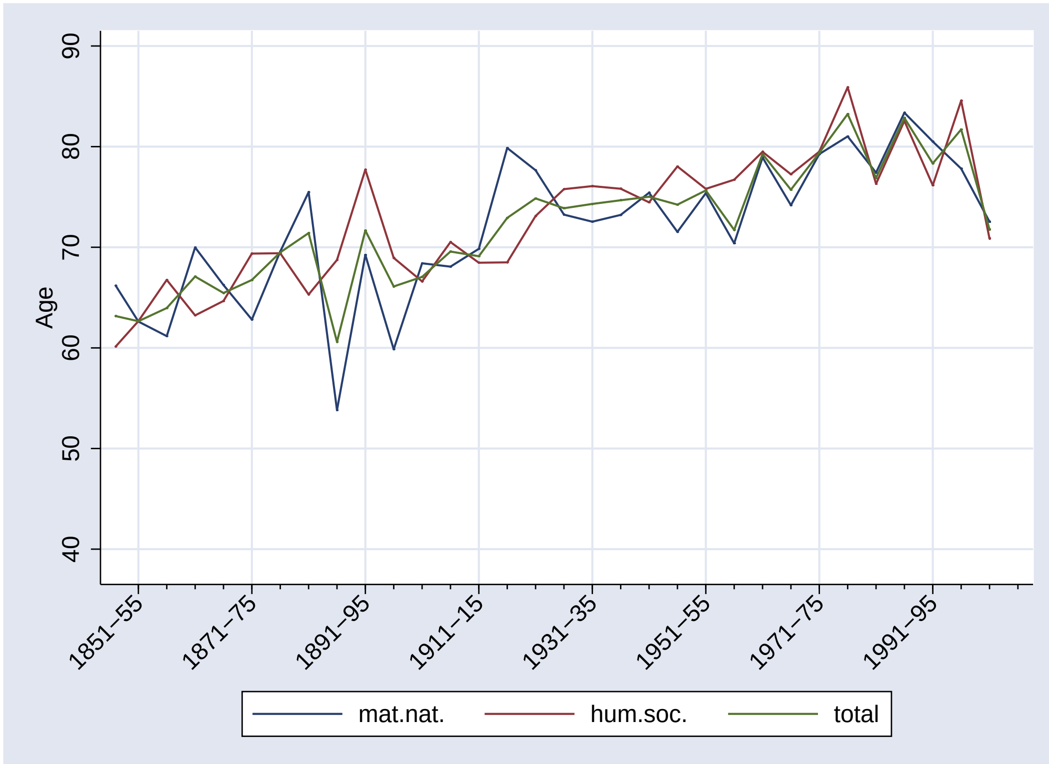

Figure 5: Mean age at death per section (5-year periods).

To reach a rejuvenation of the Academy, there exist three measures:

1. raising the number of members;
2. limiting the population of members by an upper age limit (70 years in the Austrian Academy of Sciences);
3. to elect young members.

The first "tool" has its pendant in the dynamics of large human populations which are governed by the alternative "to age or to grow". However, for various learned societies (as the Acadéemie des Sciences in France or the Austrian Academy of Sciences) this remedy is excluded due to a constant membership below a limit age (150 members below 75 in France, 90 full members below 70 in Austria). As just mentioned the second instrument is frequently used.

This leaves the third remedy as the essential steering possibility to be applied at the annual elections. There is, however, a fundamental dilemma in rejuvenating an age-structured population with constant size. Let us clarify it be a simple example which has only benchmark character.

If the Academy recruits only 47,5 year old new members, they stay 22,5 years in the system (neglecting mortality until 70 and other possibilities for exit) until the statutory retirement age. The Austrian Academy has 90 full members (45 in each section) yielding $90 : 22{,}5 = 4$ new entrants each year. If, on the other hand, only 55 year old entrants are recruited, the same calculation delivers $90 : 15 = 6$ persons.

This means that the younger the age at election, the longer the tenure in the Academy and the lower is the rate of intake. Note that a younger recruitment distribution means a younger age structure of the members measured, e.g., by the average age of the population.

It is clear that a flourishing Academy wants to recruit as many new members as possible. As already mentioned, there are new important disciplines which should be well represented in the Academy. This amounts to the central question posed in the subsequent simulations and mathematical formalization, to find an optimal trade-off between the two conflicting goals, of the Academy namely a young age structure and as many entrants as possible. This fundamental trade-off of a constant size population may be illustrated by a hyperbola as in Figure 6.

Denoting by M the total size of the body, R the member of annual intakes, and T the mean length of tenure, the stationary state is characterized by the relation

$$M = RT. \tag{1}$$

In the Austrian case we have $M = 90$, $0 \leq T \leq 30$ (assuming a minimal age at entry of 40 years and an upper age limit of 70).

The optimal election policy depends on the weights which are assigned to the conflicting goals, i.e. elect as many new members or a young age structure which implies a long duration of membership. Let us denote the weights of the former by α and of the latter by β, where $\alpha + \beta = 1$, then the objective function of the Academy is given by

$$\max V + \alpha\, R + \beta\, T. \tag{2}$$

Remark 1 Note that the relation (1) is fundamental in a stationary population where the stock equals the number of births times the life expectancy. In queuing theory which is based on birth-death processes (1) is well-know as Little's formula (see Hillier and Lieberman 1974, p. 384).

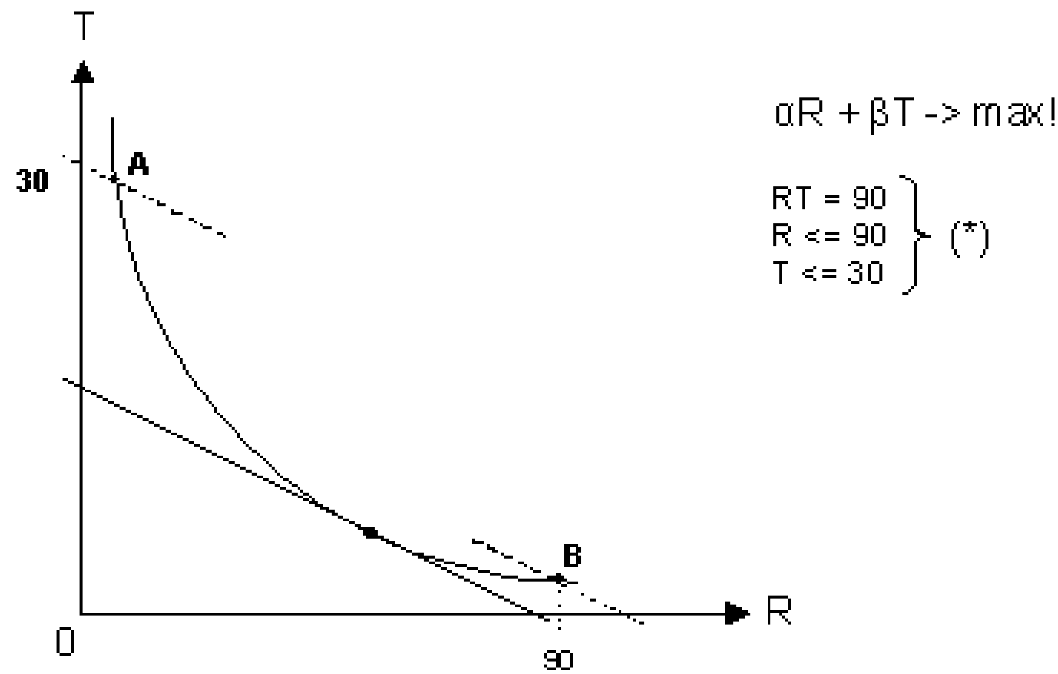

Figure 6: Trade-off between recruitment R and average tenure T

Note that the dilemma has been formulated in the important contribution of Leridon (2004) describing the demography of the French Academy of Sciences. He puts it in that way: "To counteract the spontaneous trends in ageing in the institution new members would have to be elected at increasingly young ages year after year, which would have the drawback of reducing the rate of population replacement".

To study the implication of the age distribution of new entries for the number and structure of the members of the Austrian Academy of Sciences we apply demographic projection methods. The development of mortality is assumed according to the latest forecasts of Statistics AustriaHanika and Klotz (2005), where we adjust for the lower mortality of the members of the Academy compared to the Austrian total population. We consider four alternatives scenarios of the age distribution of new entries (see Figure 7).

Status quo: Average age distribution observed during the last 25 years.

Young: Only persons below age 55 are elected.

Old: Only persons aged 55 and above are elected.

Bimodal: Persons between age 40 to 49 and 60 to 69 are elected.

The resulting number of members, number of vacancies and population that is below age 70 are plotted in Figures 8 to 10. As these figures indicate, the dilemma between a "young" society that only allows a small number of vacancies each year and a small number of the total stock of members vs. an "old" society that allows more vacancies each year and a larger stock of members each year becomes obvious. The bimodal distribution of the age at entry seems to be a compromise and as we will show in the subsequent section it also constitutes an optimal recruitment policy if the Academy's goal is to keep its member structure young and to allow a maximum number of recruitments each period.

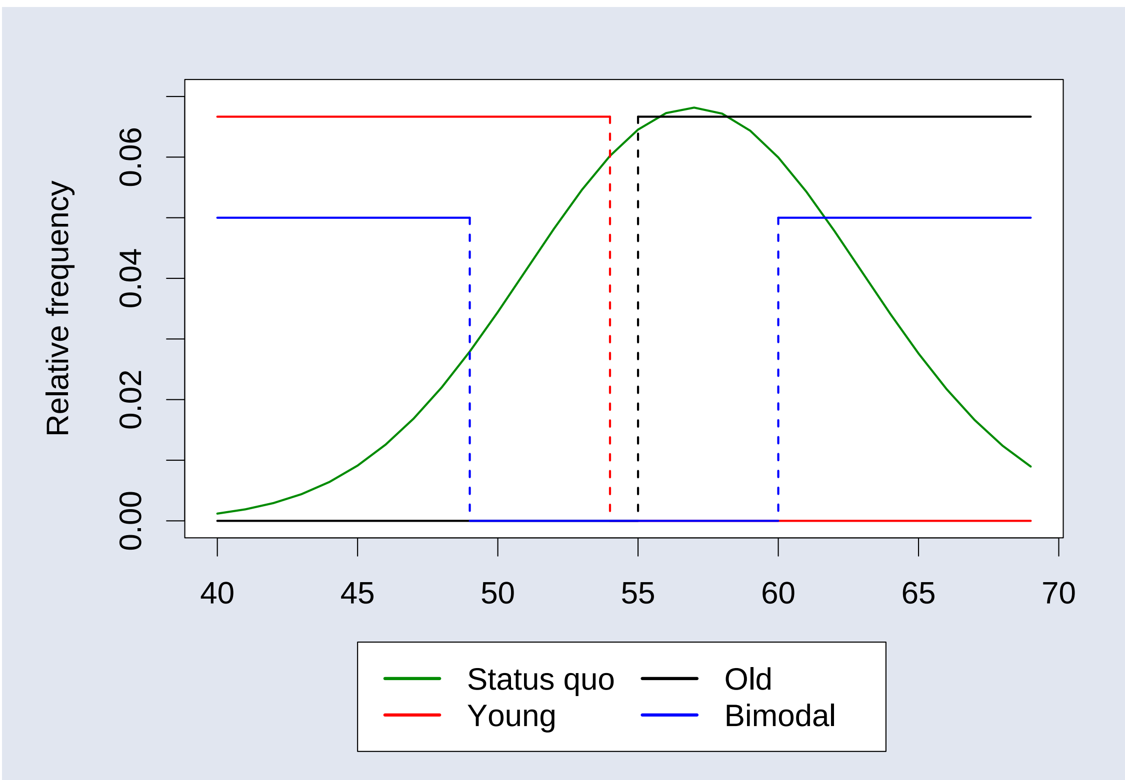

Figure 7: Alternative projection scenarios of the age distribution of election.

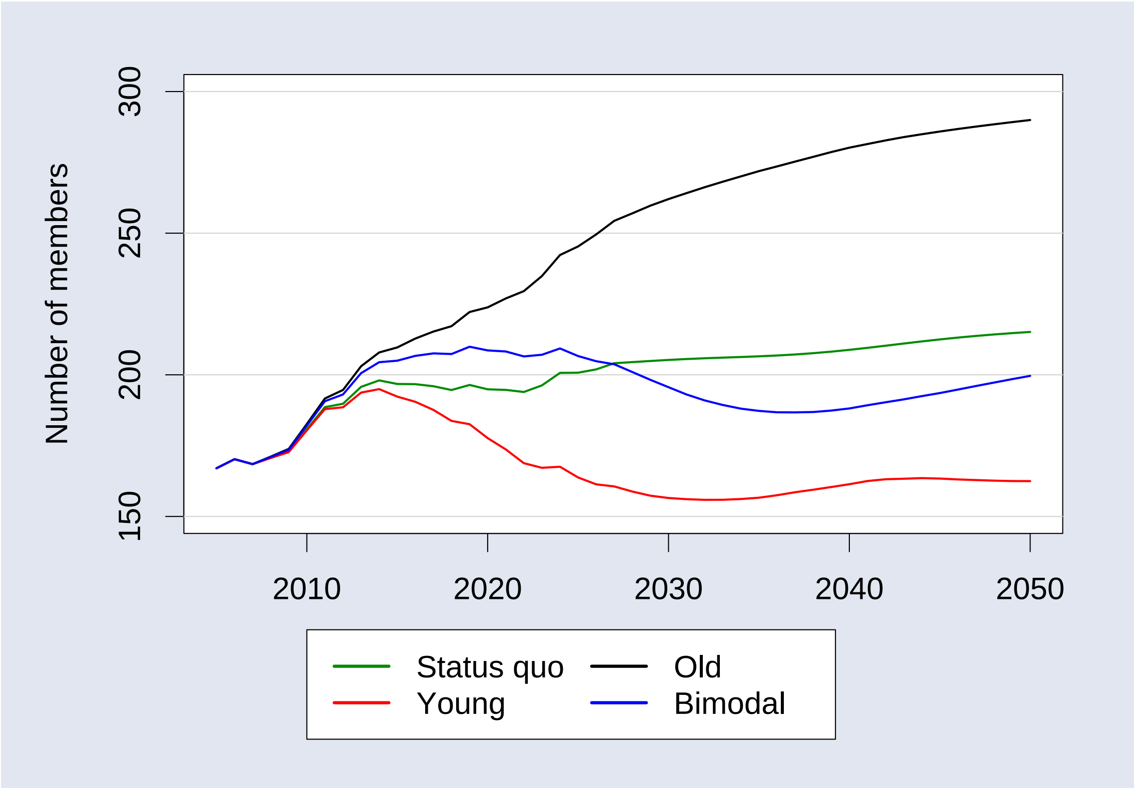

Figure 8: Projected number of members (both sections together).

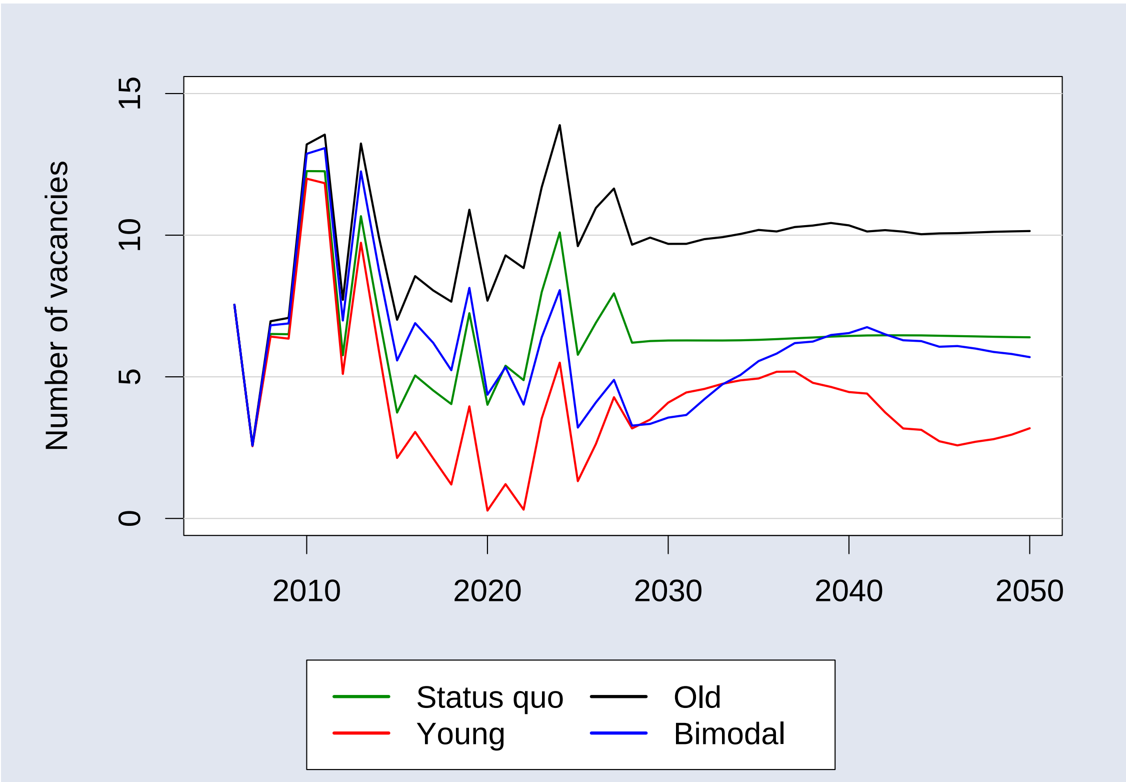

Figure 9: Projected number of vacancies (both sections together).

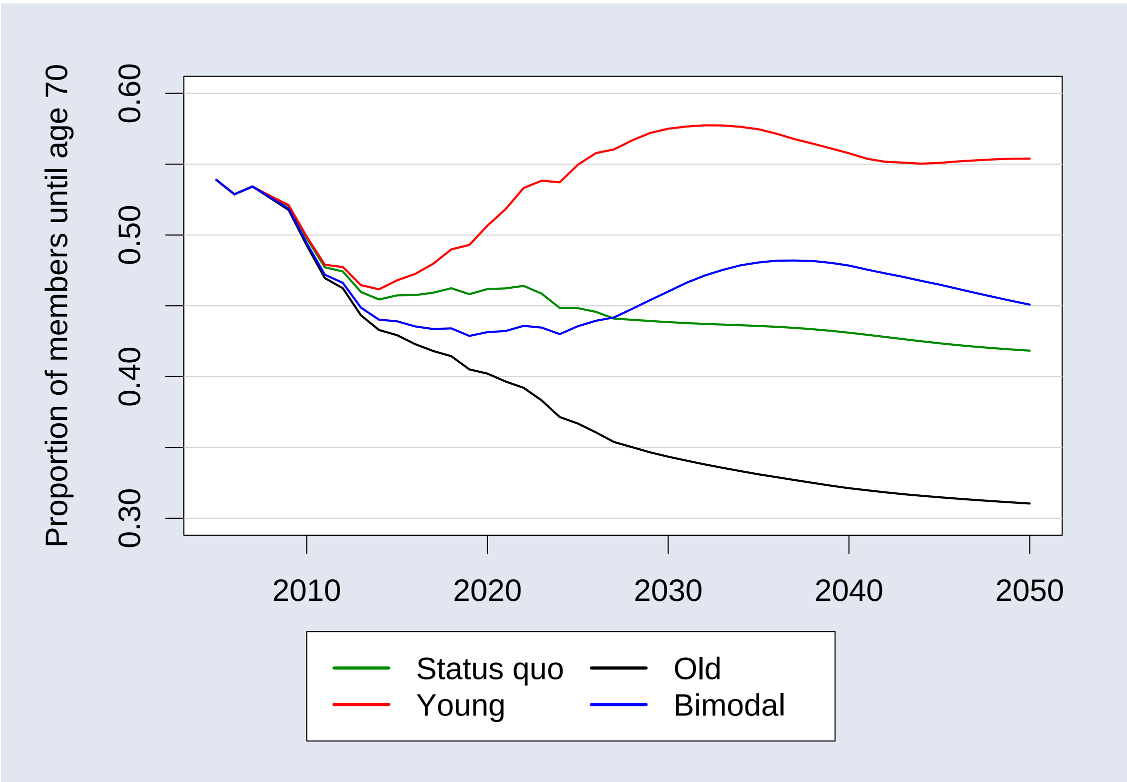

Figure 10: Projected proportion of members aged less or equal to 70 (both sections together).

4 Optimal recruitment policies

In this section we consider the issue of optimal recruitment with respect to given objectives in a learned society with a fixed size. As one objective, which is to be maximized, we consider the number of recruitments per unit of time[3]. As additional second objective one may consider characteristics of the age-structure of the learned society, for example the deviation from a given age-distribution, the ratio "old/young", the average age, etc. Therefore, we present the basic theoretical result (the conditions for optimality) for a general second objective, while the rest of the results concern the most interesting case where in addition to keeping the recruitment intensity high, the learned society seeks keeping the average age low. Several proofs are rather technical and are given in the forthcoming paper Feichtinger and Veliov (n.d.).

4.1 Formalization of the problem and optimality conditions

The dynamics of populations of fixed size plays an important role in demography (e.g. migration to guarantee zero population growth for below-replacement fertility) and in man-power planning, see e.g. Preston (1970), Feichtinger and Mehlmann (1976), Mitra (1983), Arthur and Espenshade (1988), G. and Steinmann (1992), Schmertmann (1992). The following model for the dynamics of the age-structure involves a non-standard version of the McKendrick equation (McKendrick 1926, Webb 1985, Aniţa 2000):

$$M_t(t,a) + M_a(t,a) = -\mu(a)M(t,a) + R(t)u(t,a), \tag{3}$$

$$R(t) = M(t,\omega) + \int_0^\omega \mu(a)M(t,a)\,\mathrm{d}a, \tag{4}$$

with the side conditions

$$M(0,a) = M_0(a), \quad M(t,0) = 0. \tag{5}$$

Here
$M(t,\cdot)$ is the age density of the members of the society at time t;
$\mu(a)$ is the mortality rate of the members at time t and age a;
$R(t)$ is the intensity of recruitment at time t;
$u(t,\cdot)$ is the age density[4] of recruitment at time t;

[3]For an Academy of Sciences, for example, a too small number of elections would frustrate the scientists outside the academy and would decrease the stimulative role of the academy. Moreover, the new members bring new ideas, represent new areas, etc.

[4]To avoid misunderstanding we stress that $M(t,\cdot)$ needs not be a probabilistic density, while $u(t,\cdot)$ is assumed to be a probabilistic density, in the sense given by the equality in (6) below.

$M_0(\cdot)$ is the initial age-density of members;
ω is a fixed exit (retirement) age of members;
$M_t + M_a$ is the sum of the partial derivatives of M (strictly speaking, this is the derivative of M in the direction $(1,1)$ in the (t,a)-plane, i.e. the change along a diagonal in the Lexis diagram).

The dynamics of the age structure of the learned society is given by the classical McKendrick equation (3), while (4) indicates that the size of the organization is fixed and equals $\bar{M} = \int_0^\omega M_0(a)\,\mathrm{d}a$ (this can be easily seen by integrating (3) over a and utilizing the assumption for fixed size. Alternatively (4) can be understood as follows: At any time point t the intensity of recruitment $R(t)$ is determined by the number of people reaching the threshold age ω (first term on the r.h.s.) and the number of deaths where the latter is determined by the sum of age specific deaths (second term on the r.h.s).

The following constraints are posed for the recruitment density, $u(t,\cdot)$, which is considered further as a control (decision) variable:

$$0 \le u(t,a) \le \bar{u}(a), \quad \int_0^\omega u(t,a)\,\mathrm{d}a = 1. \tag{6}$$

The upper bound, $\bar{u}(a)$, for the control has a different meaning in different practical situations and will be discussed in Section 4.3 for the case of an Academy of Sciences.

As mentioned above, we focus our analysis on two objectives that are to be maximized:
– the recruitment intensity, $R(t)$.
– a general objective of the form $\int_0^\omega A(a, M(t,a), u(t,a))\,\mathrm{d}a$ depending on the densities of the members and of the recruitment.

The average age of the members is a special case, where $A(a,M,u) = -aM/\bar{M}$ (taken with minus sign since it is to be minimized), or merely $A(a,M,u) = -aM$.

Since more than one objective are involved, we employ the Pareto optimization framework, considering the aggregated objective function

$$\max \int_0^\infty e^{-rt} \left[\alpha R(t) + \beta \int_0^\omega A(a, M(t,a), u(t,a))\,\mathrm{d}a\right] \mathrm{d}t, \tag{7}$$

where $r > 0$ is a time-preference rate, $\alpha \ge 0$ and $\beta \ge 0$ are weights attributed to the two objectives.

Although the above problem is a rather specific one, it provides a number of challenges: (i) it is nonlinear (although bilinear); (ii) the time horizon is infinite, which creates substantial difficulties in obtaining appropriate transversality conditions for the co-state system[5]; (iii) due to equation (4) the dynamics in (3) is non-local;

Further we denote for brevity $\Omega = [0, \omega]$, $D = [0, \infty) \times [0, \omega]$. Below it will be assumed that the exogenous data μ, $\bar{u}$, and M_0, are non-negative and piece-wise continuous, and the integrand in the objective function $A = A(a, M, u)$ is continuous, concave in (M, u), differentiable in M, and the derivative is Lipschitz continuous. Moreover, it is assumed that $\int_\Omega \bar{u}(a)\,\mathrm{d}a \geq 1$, since otherwise the two requirements in (6) would be contradictory.

For the notion of a solution to the system (3)–(5) we refer to the monographs Webb (1985), Aniţa (2000) and G. Feichtinger and Veliov (2003).

Lemma 1 *For every piecewise continuous function u satisfying (6), the system (3)–(5) has a unique solution in D, and the solution is bounded.*

Proposition 1 *Problem (3)–(7) has a solution.*

The optimal solution is not unique, in general. However, later on uniqueness will be established under an additional assumption.

Problem (3)–(7) is on infinite horizon, and involves the "advanced" term $M(t, \omega)$ in the right-hand side of the differential equation. For each of these two reasons the general optimality conditions (maximum principle) obtained in Brokate (1985), G. Feichtinger and Veliov (2003), or the other known optimality conditions for McKendrick-type control systems (see the references in G. Feichtinger and Veliov (2003) and footnote 5) are not applicable.

To obtain a necessary optimality condition we introduce the following *adjoint system* for given reference measurable and bounded functions M and u:

$$\lambda_t(t, a) + \lambda_a(t, a) = (r + \mu(a))\lambda(t, a) - \mu(a)\eta(t) - \beta A_M(a, M(t, a), u(t, a)), \quad (8)$$

$$\eta(t) = \alpha + \int_0^\omega \lambda(t, a) u(t, a)\,\mathrm{d}a, \quad (9)$$

with the boundary condition

$$\lambda(t, \omega) = \eta(t). \quad (10)$$

[5] Out of many papers that consider optimal control of McKendrick equations, only Luo et al. (2003) studies the asymptotic behaviour of the adjoint variable for a special system (not including the problem considered here), but the transversality condition obtained there is based on an implicit assumption, the verification of which is, actually, the main trouble. Our approach is substantially different.

Theorem 1 *(Feichtinger and Veliov n.d.) Let (u, M, R) be an optimal solution of problem (3)–(7). Then the adjoint system (8)–(10) has a unique bounded solution λ on D. Moreover for (almost) every t the optimal control, $u(t,\cdot)$, maximizes the integral*

$$\int_0^\omega [\lambda(t,a)R(t)v(a) - \beta A(a, M(t,a), v(a))] \, \mathrm{d}a$$

on the set of functions $v(\cdot)$ satisfying the constraints

$$0 \le v(a) \le \bar{u}(a), \quad \int_0^\omega v(a) \, \mathrm{d}a = 1. \tag{11}$$

Remark 2 The above adjoint system has infinitely many solutions on D. In general, certain transversality conditions at infinity are needed to ensure uniqueness. Here the role of a transversality condition is played by the requirement that the solution is bounded. This non-standard (and rather strong) transversality condition plays a key role in the proof of the above theorem and all the subsequent analysis (see Feichtinger and Veliov n.d.).

In the particular case where $A(a, M, u) = -aM$ (the average age of members is to be minimized as a second objective) the adjoint equation (8) reduces to

$$\lambda_t(t,a) + \lambda_a(t,a) = (r + \mu(a))\lambda(t,a) - \mu(a)\eta(t) + \beta a, \tag{12}$$

4.2 Properties of the optimal recruitment policy

In this subsection we obtain some qualitative consequences of the optimality conditions, focusing on two objectives: high intensity of recruitment and low average age of the members of the learned society. That is, instead of (7) we consider the problem of maximization of the functional

$$\max \int_0^\infty e^{-rt} \left[\alpha R(t) - \beta \int_0^\omega a M(t,a) \, \mathrm{d}a \right] \mathrm{d}t, \tag{13}$$

subject to the constraints (3)–(6).

Further in this section we assume that the following condition for the mortality function is fulfilled: $\mu(a)$ is a continuously differentiable and non-decreasing function, which equals zero on some interval $[0, a_0)$, and is strongly convex on $(a_0, \omega]$.[6] This assumption seems to be plausible in the age interval 40–70.

[6] In most of the results below this assumption can be substantially relaxed, as in Feichtinger and Veliov (n.d.). For example for the two theorems it is enough to assume that for every real numbers $d > 0$ and e the set of solutions of the equation $\mu(a) + dp(a) = e$ is of zero Lebesgue measure.

Theorem 2 *(Feichtinger and Veliov n.d.) The optimal control for problem (13), (3)–(6) is unique, time-invariant (that is, $u(t,a) = u(a)$), and satisfies the following conditions:*

$$\int_\Omega \xi(a)u(a)\,\mathrm{d}a = \max_v \int_\Omega \xi(a)v(a)\,\mathrm{d}a, \tag{14}$$

subject to (11) and the additional condition

$$\int_\Omega \xi(a)v(a)\,\mathrm{d}a = -\alpha, \tag{15}$$

where $\xi = \xi(a)$ is the solution of the equation

$$\xi' = (r + \mu(a))\xi - \beta a + \nu, \quad \xi(\omega) = 0, \tag{16}$$

and ν is a "free" parameter.

In fact, the free parameter, ν, in the above formulation should be determined in such a way, that the solution of (14), (11) (which can be proved to be unique on the regularity assumption) for this value of ν satisfies also the equality (15) (with ξ solving (16)). Since $\xi(a)$ represents the "shadow price" of members of age a (normalized in such a way that it equals zero at age ω), the integral in the right-hand side of (14) is the "value" of the recruitment density v, which is to be maximized.

Remark 3 The assumption for the mortality rate made above is essential for the above theorem (although it can be relaxed, see footnote 6). The optimal control is not unique, in general. Moreover, the optimality condition in the above theorem is not sufficient, in general.

Theorem 2 implies that for any given initial distribution $M_0(a)$ of the members, the optimal recruitment policy u does not change with time. The next (*strong ergodicity*[7]) result asserts that the optimal recruitment distribution does even not depend on the initial distribution of the members.

Theorem 3 *(Feichtinger and Veliov (n.d.)) The unique optimal control $u(t,a) = u(a)$ in problem (13), (3)–(6) is independent of the initial data $M_0(\cdot)$ (satisfying the balance equality $\int_\Omega M_0(a)\,\mathrm{d}a = \bar{M}$). For the optimal path $M(t,a)$ corresponding to the control u and any initial distribution M_0 it holds that*

$$\lim_{t\to\infty} \sup_{a\in\Omega} |M(t,a) - M^\infty(a)| = 0,$$

[7] see e.g. Cohen (1979), Arthur. (1982) for definition of ergodicity in demographic context.

where $M^\infty(a)$ is given by the following formula:

$$M^\infty(a) = \frac{\int_0^a u(s)\psi(a|s)\,\mathrm{d}s}{\int_0^\omega u(s)\int_s^\omega \psi(a|s)\,\mathrm{d}a\,\mathrm{d}s}\bar{M},$$

and

$$\psi(a|s) = e^{-\int_s^a \mu(\theta)\,\mathrm{d}\theta}$$

is the conditional survival probability of an individual till age a, provided that the individual is alive at age $s \leq a$.

This means that following the optimal (time-invariant) recruitment policy, the optimal distribution path converges to the stationary distribution M^∞, which is independent of the initial data M_0. According to the above definition, the stationary distribution can be understood as follows: to obtain the number of members at age a the numerator represents the sum of all members recruited at age $s < a$ surviving to age a. The denominator gives the sum of all recruits, i.e. the sum over all ages of recruits $u(s)$ surviving to the threshold age ω.

Remark 4 Although Theorem 2 and 3 assert that the optimal recruitment policy is independent of time and of the initial $M_0(a)$ of the members, the number of recruits, $R(t)$, may depend on the time and on the initial distribution M_0.

Remark 5 The strong ergodicity property is not valid, in general, for the entirely discrete version of the problem considered in this paper. It seems that the strong ergodicity of the optimal solution does not hold also in the continuous case, unless an appropriate assumption, such as the assumption for the mortality rate, is posed.

One may consider the steady-state optimization problem

$$\max_{u(\cdot),R}\left\{\alpha R - \beta\int_0^\omega aM^\infty(a)\,\mathrm{d}a\right\},$$

subject to (6) and the state constraint $\int_\Omega M^\infty(a)\,\mathrm{d}a = \bar{M}$. We stress that the optimal control u^∞ for this problem does not coincide with the optimal control u of the intertemporal optimization problem (13), (3)–(6). The latter depends on the time-preference rate r, while u^∞ is independent of r. Higher values of r turn out to encourage recruitment of more young members. The intuitive explanation is that recruiting younger members has an immediate effect on the average age of the academy, while the negative effect on the recruitment intensity (due to the longer membership till exit) appears later.

Now we shall analyze in more details the properties of the optimal recruitment policy $u(a)$ making use of Theorem 2.

Proposition 2 *(Feichtinger and Veliov n.d.,* Bi-polar recruitment principle) *The optimal recruitment control, $u(a)$, has the following structure: there are numbers $0 \le \theta < \tau < \omega$ such that*

$$u(a) = \begin{cases} \bar{u}(a) & \text{for } a \in [0,\theta) \cup (\tau,\omega], \\ 0 & \text{for a} \in [\theta,\tau]. \end{cases} \tag{17}$$

The proof can be found in the appendix.

Remark 6 The natural mortality rate in the ages above 30 satisfies the assumptions for μ (in fact, with $a_0 = 0$, but in many considerations one may neglect the mortality in young ages, therefore we formulate the above more flexible assumptions for μ). A remarkable fact is, that for $\alpha > 0$ the second interval of recruitment is always non-degenerate: $\tau < \omega$.

We stress that in practical applications the principle of bi-polar recruitment should not be taken in absolute sense, as far as usually many other criteria (different than the recruitment intensity and the average age of the members of the learned society) are taken into account. The essence of the result is, that if the last mentioned two criteria matter for the society (which is, indeed, the case for many organizations as academies of sciences or academies of awards), then they have a polarizing effect on the optimal recruitment policy: they shift the recruitment partly to younger and partly to older ages, decreasing in this way the middle-age recruitment. An interesting interpretation of this result is given by Warren Sanderson[8]. The essence is, that an academy of awards should focus on awarding relatively young talents for recent outstanding achievements, and, on the other hand, old persons for their all-life-long contributions. More discussion about the practical implementation of the principle follows in the next subsection.

[8]Warren Sanderson, Professor of Economics, SUNY-Stony Brook, Stony Brook, N.Y. 11794-4384, USA. Personal communications with the authors.

4.3 A case study: the Austrian Academy of Sciences

The size of the Austrian Academy of Science (AAS) is fixed: $\bar{M} = 90$. Members of age above 70 are not counted, members below 40 are rather seldom exceptions, therefore we have $\omega = 30$ (which corresponds to the age interval 40–70 years). The age-specific mortality rates are identified from data of the Austrian Academy of Sciences (AAS) as reviewed in section 2.

Before presenting some numerical calculations for AAS we give one interpretation of the upper bound $\bar{u}(a)$ for the recruitment density $u(a)$, which influences the quantitative results. The election decision in an Academy is based on the judgment of the members, based primarily on the quality of the candidates in terms of accumulated contributions, current activity, expected future contributions related to the goals the Academy, etc. Let $u^*(a)$ be the statistical normalized age-density of elections in the Academy. If the Academy is concerned by its average age and/or number of elections per year, then current election practice, $u^*(a)$, has to be modified, taking into account also the formal objectives in (13). A reasonable approach to do this is to consider the optimization problem (13), (3)–(6) with

$$\bar{u}(a) = \sigma u^*(a),$$

where $\sigma \geq 1$ is a "tolerance" factor. If $\sigma = 1$, then $u(a) = u^*(a)$ would be the only admissible policy (due to the constraint (6)). The higher is the value of σ, the more is the current practice sacrificed in favor of the formal objective (13). In what follows we fix the moderate value $\sigma = 2$. Moreover, the choice of $\bar{u}(a)$ may reflect some administrative regulations. Below we assume $\bar{u}(a) = 0$ for $a > 69$, that is, no elections are allowed at age above 69 years.

The "decision-maker" has to specify also the weights α and β of the two objectives in (13), where it may be assumed that $\alpha + \beta = 1$. Choosing $\alpha = 1$ one takes into account only the recruitment intensity, disregarding the average age. In this case the optimal recruitment is totally shifted to older ages: $\theta = 0$ in the representation (17) of the solution. The resulting number of elections at the optimal steady state for AAS is $R^* = 10.7$, with corresponding average age of the members $A^* = 64.9$. Notice that with an infinite tolerance factor σ, one would have infinite number of elections with average age 70 years.

With $\beta = 1$ one disregards the number of elections, which results in average age $A_* = 60.7$ years with $R_* = 5.4$ elections per year. For an infinite tolerance factor this numbers would be $A = 55.0$ years, and $R = 3.0$.

Figure 11 represents the optimal election policy (the solid line) for a choice of the weights which results in $R = 7$ elections per year at the steady-state. This number is chosen since it corresponds to the present practice in the AAS. That is, a minimization of the average age is sought (given the tolerance factor $\sigma = 2$), keeping the the number of elections per year at the present level. The dash-dotted line shows the upper bound for the control, $\bar{u}$. The dotted line represents the function $\xi(a) - \nu$ (re-scaled to fit to the same plot) which is to be maximized by the optimal control $u(a)$, according to the maximum principle (14). This function plays the role of a "shadow price" of members. It has lowest values in the central part of the age interval $[0, 30]$ (corresponding to 40–70 years of real age), where there are no elections. The election are concentrated in young and in old ages as the bi-polar recruitment principle suggests. The average age of the academy members is $A = 60.95$. With the present election practice at AAS the average age is 62.0 years. The improvement is moderate (about one year lower average age), which is due to the relatively low tolerance factor. With a tolerance factor $\sigma = 5$, for example, the average age with $R = 7$ elections per year would be 59.4 years.

The qualitative essence of the result is: if a certain compromise with the existing election criteria would be tolerable, aiming at reduction of the average age while keeping the present number of elections and the size of the academy the same, then the optimal way to use this compromise is to shift some of the middle-age elections (50–60 years) to younger ages (40–50 years) and to older ages (60–69 years).

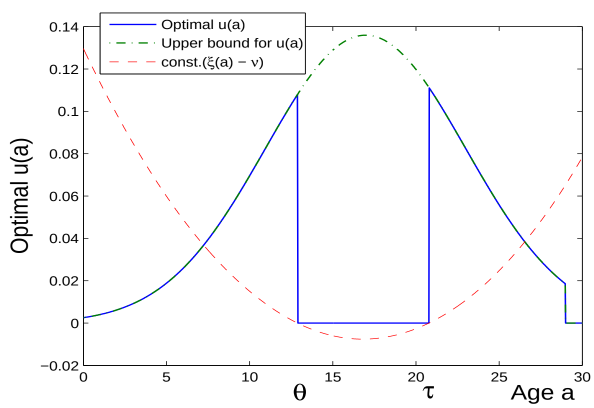

Figure 11: The optimal recruitment density $u(a)$, the upper bound $\bar{u}(a)$, and the switching function $\xi(a) - \nu$.

The last issue we address is the influence of mortality on the optimal election policy. The mortality rate of the members of the AAS is lower than for the total population and its rate per year reaches 0.01216 for the ages between 65 and 70. However, below we show that the effect of the mortality on the average age of the members and on the optimal election policy is not negligible. We know already that for $R = 7$ elections per year the optimal average age of the members is 60.95, while if zero mortality is assumed in the age range $[40, 70]$, the optimal average age is 61.35. The presently existing mortality "contributes" to a younger academy, which is to be expected, but the difference in the average ages is not big. However, the dependence of the optimal election policy on the mortality is considerable. Indeed, we consider the "young-to-old ratio", which is the ratio of the number of elections in the young years (in the interval $40+[0, \theta]$) to that in the old years ($40+[\tau, 30]$). This ratio is approximately 1.02 for the optimal undiscounted control with the present mortality at the AAS, while it shifts to 0.75 if the mortality is neglected. Somewhat surprisingly, in the case of zero mortality the optimal election policy shifts to older ages. The reason for this effect is that more elections at older ages increase the intensity of exits due to maximal age ω (70 years in real age), which is the only reason for exits if the mortality is zero. In a sense, the higher elections at old ages in the case of zero mortality is a substitution for the missing exits due to mortality.

5 Conclusions

In this study we investigated the age dynamics of learned societies. Learned societies, like the Austrian Academy of Sciences, are mostly of constant population size as defined by its bye-laws. Hence, the number of new members is determined by the number of exists from the reference population either by death or by surpassing a statutory retirement age. Over the recent decades, many learned societies including the Austrian Academy of Sciences, faced a considerable ageing of its member population. By using projection methods, we study alternative scenarios for the future development of the age structure of the members of the Austrian Academy of Sciences. The projections reveal a trade-off between the annual number of vacancies and the age at election of the members. When Leridon (2004) did a similar study on the French Academy of Sciences, he summarized: "To counteract the spontaneous trends in ageing in the institution new members would have to be elected at increasingly young ages year after year, which would have the drawback of reducing the rate of population replacement."

In this context our goal was to find an optimal trade-off between a young age structure of a learned society and a large recruitment rate of excellent researchers. To study these conflicting goals distributed parameter control age methods have been applied. The underlying age dynamics is the McKendrick partial differential equation (see McKendrick 1926). We derive an optimal recruitment policy which is bi-polar, i.,e. a shift of the recruitment partly to younger ages and partly to older ages, decreasing in this way the middle-age recruitment. Such a recruitment policy can be interpreted as on the one hand, awarding relatively young talents for recent outstanding achievements and on the other hand, older scientists for the life-long contributions.

The presented analysis may not be limited to learned societies. There are various further applications of optimal control problems for age-structured (or/and duration-dependent) populations in demography and other social and life sciences. Let us briefly mention some of then (without an attempt to be complete):

Migration: Immigrant's ages and the structure of stationary population with below-replacement fertility have important policy implications. Schmertmann (1992) provides an interesting discussion of the problem.

Manpower planning: Young and Almond models, Feichtinger and Mehlmann (1976), Preston (1970).

Epidemiology: Efficient control of the HIV/AIDS pandemic, optimal particularly trade-off of medical treatment (oral vaccination) and prevention.

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APPENDIX

Proof of Proposition 2

Proof. We apply the characterization of the optimal control given in Theorem 2. The regularity assumption is apparently fulfilled, since on the additional assumption of the proposition the function $\mu(a) + dp(a) = e$ may have at most two zeros.

Since ξ is twice continuously differentiable, all we have to prove is, that ξ has no local maxima in $(0, \omega)$.

Assume that $a \in (0, \omega)$ is a local maximizer of ξ, and $\xi(a) \geq 0$. Then

$$\xi''(a) = \mu'(a)\xi(a) + \beta > 0,$$

which is a contradiction. Now assume that $\xi(a) < 0$. Since $\xi(\omega) = 0$, there must be a local minimizer $b \in [a, \omega)$ with $\xi(b) \leq \xi(a)$. Then

$$0 \geq \xi''(a) = \mu'(a)\xi(a) + \beta \geq \mu'(b)\xi(a) + \beta \geq \mu'(b)\xi(b) + \beta = \xi''(b) \geq 0.$$

Then all the inequalities must be equalities. Since $\xi(a) < 0$, this implies $\mu'(a) = \mu'(b)$, which yields $a \in (0, a_0]$. Since μ is identically zero on $[0, a_0)$, we have $\xi''(a) = \beta$, which is a contradiction. Thus the optimal $u(a)$ has the structure (17). It remains only to prove that $\tau < \omega$. If this is not the case, then $\xi(a) \geq 0$ for all a, for which $u(a) > 0$. This contradicts (15). Q.E.D.