Financial demography

Mastering the financial consequences of life contingencies*

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Abstract

In 1982, Nathan Keyfitz and Andrei Rogers published contingency calculations that bridged the historical divide between demography and actuarial sciences and paved the way to a better management of financial risks at the level of cohorts and individuals. Many events in life involve a financial risk. Financial security requires an understanding of the life course and an assessment of the financial consequences of life events. This paper extends the work by Keyfitz and Rogers and presents a biographic actuarial model that combines *life history models*, developed in demography, sociology, health sciences and actuarial sciences, with a large variety of *insurance schemes*. The model is generic. It is not restricted to a particular type of insurance or financial protection. The life course is approached as a sequence of state occupancies and state transitions (events), and is modelled by a discrete-state continuous-time Markov process in two time scales: individual time (age) and historical or calendar time. Premiums and insurance benefits are linked to transitions and state occupancies. The equivalence principle is used to determine premiums that cover insurance benefits in complex life histories. An example from disability insurance illustrates the model. In an era when the public sector and the private sector generate innovative schemes and products aimed at financial protection throughout the life course in the presence of multiple demographic contingencies, a synthesis of demographic and actuarial modelling presents a potentially major new window of opportunities. Financial demography is a new field that responds to the challenges that contemporary and envisaged demographic changes cause in public finance and personal finance.

1. Introduction

Many decisions we make in life and many events we experience involve the risk of a loss. Frequently, the loss is financial. When it is substantial, it affects our lifestyle and quality of life. The decision to leave employment for raisons of retirement, family reasons or to return to school involves a financial risk. Some events have limited financial consequences such as the loss of a mobile phone or the theft of a bicycle. Other events such as a theft of a car, the loss of a house due to a fire or a flood have significant consequences. Serious financial consequences are associated with the death of the provider of family income, loss of a job or a serious health problem. Events with serious consequences that may extend over the entire remaining lifetime are generally referred to as life events, a term borrowed from psychology and epidemiology (Goldberg and Comstock, 1980). In insurance, the term *contingencies* is used. Contingencies are random events that have major impacts when they occur. They include life contingencies, disaster contingencies, disease contingencies, etc.. Risk is defined as the financial consequence of an unforeseen event implying a loss (Frees, 1998, p. 34).

To limit the financial consequences of a decision or an event, i.e. to remain in a secure financial position throughout the life course, risk averse and rational individuals reallocate financial resources (e.g. income) over the life course accounting for uncertainties and contingencies, and engage in risk sharing by transferring risks to other individuals or institutions and by paying a premium to compensate the one who takes over the risk. Risk sharing mechanisms transform individual risks into collective risks that are easier to tame. Traditional risk sharing schemes relied on mutual aid. Modern schemes rely on pooling financial resources and cost sharing. Risks may be shared by a select group of people (limited coverage, voluntary insurance) or by the society at large (mandatory or compulsory insurance). In many insurance schemes, risk sharing involves the public sector and the private sector. Government programmes that provide assistance to persons faced with unemployment, disability or health problems are part of social security or social insurance[1]. The method presented in this paper is a general one because of its focus on the human life course and the associated contingencies. The method is applicable in private insurance and social insurance, and to life and non-life insurance.

Financial security during the life course calls for risk management. Risk management involves the identification of unwanted events, preventive strategies that reduce the likelihood of unwanted events and insurance against losses incurred once an unwanted event occurs. In principle, insurance can be associated with any contingency in life. In this paper no distinction is made between types of contingencies and types of insurers. Life course risk management is based on the premise that events in life are predictable and the financial consequences can be quantified. That requires an ability to determine for every event at least three measures: the likelihood of occurrence, the timing of

[1] Public sector institutions that organize and manage social security and social insurance, including programs for pensions, disability, survivor and unemployment insurance, and medical expenditures are referred to as the welfare state (de Mooij, 2006, p. 41).

occurrence (age at event) and the financial loss incurred. The loss may be immediate, such as in the case of a car theft, or it may be spread out over time, such as in the case of retirement or the loss of a job. A loss that is distributed in time is likely to depend on the subsequent lifepath, which adds uncertainty.

Financial protection involves arrangements for a reallocation of funds from one stage of life to another and for transfers of funds from persons who do not experience income loss or excessive expenditures to persons who do. The effect of the arrangements is a smoothing of the income and expenditures over the life course and between members of a group or population. Arrangements can be individual or collective. The reallocation of income over the life cycle is predominantly an individual arrangement but collective schemes are significant too. For instanced, studies show that social security and taxation schemes offer more financial protection by life-cycle smoothing than by redistributing income between individuals (Falkingham et al., 1993; de Mooij, 2006, p. 124)[2]. Group membership and entitlements are generally based on personal attributes and preferences. Because of the heterogeneity within a group, collective arrangements may involve substantial redistribution of funds within a group and, as a consequence, may cause substantial equity issues. For instance, pension schemes in which currently employed persons collectively pay for the pensions of currently retired persons tend to redistribute funds from the poor to the rich because the rich generally live longer (de Mooij, 2006, p. 119; Caselli et al., 2003). Disability insurance redistributes funds in the opposite direction because the poor are more prone to health problems and disability. Equity issues are central to insurance schemes (risk sharing mechanisms) that involve risk classification, i.e. the classification of individuals into groups on the basis of risk levels (see e.g. Cummins et al., 1983). Equity issues are also central to social protection schemes that extend over different birth cohorts or generations and that may result in substantial intergenerational transfers.

The purpose of this paper is to present a model that describes the financial lifepaths of cohorts and individual cohort members in the presence of individual and collective arrangements. The model consists of two modules. The first is a demographic model that describes and projects the life courses of cohorts and cohort members. It differs from traditional demographic projection models in the level of detail about the life course. Traditional models project a population by age and sex and consider only the events of birth, death and migration. The demographic model presented in this paper is a generic model that considers several life events or contingencies in the life course. Because of its focus on the individual life course, it is referred to as a *biographic* model rather than a *demographic* model. In the biographic model, the life course is operationalized as a sequence of events and a sequence of states. Events are transitions between the states. The biographic model describes and projects cohort biographies and individual biographies in terms of states occupied and transitions experienced. The second module is an *actuarial* model that associates payments with events or transitions and with state occupancies and determines the actuarial value (expected present value) of a single

[2] A recent study by the Netherlands Bureau for Economic Policy Analysis suggests that between 60 and 80 percent of the welfare state actually concerns intrapersonal reallocation of income over the life cycle, rather than redistribution between rich and poor (de Mooij, 2006, p. 137).

payment or a series of payments. The biographic actuarial model is an instrument for financial life planning at the individual level and the cohort level. It can effectively be used to determine the transfers between stages of life and between members of a cohort that are necessary for financial security throughout the life course. The model can easily be extended to multiple cohorts and used to quantify intergenerational transfers in payment schemes that involve several cohorts such as in the PAYGO pension scheme and the more recent notional defined contribution (NDC) schemes.

The biographic actuarial model is a multistate probability model, more specifically a continuous-time Markov chain. A multistate model distinguishes several functional states in one or different domains of life. State variables denote attributes of individuals; they include employment status, level of education, marital status, family status, health status, region of residence, etc.. The state an individual occupies at a given age and the transition from one state to another cannot be predicted with certainty and therefore depend on chance. They are represented by random variables. Sequences of random variables are stochastic processes that can be described by stochastic models (see e.g. Taylor and Karlin, 1994). A continuous-time model is chosen because life events and other transitions between functional states are not restricted to discrete times. Transitions occur in continuous time. They are governed by transition intensities that may depend on a range of factors (covariates, situational variables, etc.). Systematic factors determine the expected transition intensities for groups of individuals with the same characteristics. Distributions around the expected values determine the individual variations. The transition intensities represent the fundamental parameters of the biographic model. They are often approximated by transition rates and transition probabilities and they are generally estimated from observations on individual life histories. For a non-technical illustration of the biographic model, see Willekens et al. (2006).

In actuarial sciences and demography, multistate models have a relatively long history. Among the first uses of Markov chain models in life contingencies and their extensions were by Amsler (1968), Hoem (1969), Consael and Sonnenschein (1978), Keyfitz and Rogers (1982), Waters (1984, 1989) and Ramlau-Hansen (1988). Hoem (1988) presents a first comprehensive discussion of how life insurance mathematics can be embedded in the theory of Markov chains. Keyfitz and Rogers introduce multistate demographic models into actuarial sciences. The first textbook on Markov models in (life) insurance was published by Wolthuis (1994; second edition 2003). Another textbook is Haberman and Pitacco (1999). Ramlau-Hansen (1988) introduces the counting process framework in life insurance. For a review of the early literature, see Hoem (1988) and Jones (1993, 1994). Wolthuis (2003) developed a model for life insurance, Pitacco (1995) and Haberman and Pitacco (1999) and Cordeiro (200) developed models for disability insurance, and Gritz et al. (1998) and Debicka (2005) for unemployment insurance. Attema (1997) elaborated the multistate Gompertz model. The actuarial model proposed in this paper is a general model that encompasses these models. Multistate actuarial models have much in common with multistate models developed and used in demography and other sciences. In demography, multistate models have a long history too and they are at the origin of the subfield of multistate demography (for a brief historical overview, see Willekens, 2003). Multistate demographic models were developed by Rogers (1975), Schoen (1988) and others to estimate and project (1) the numbers of individuals in stages of life or functional states at a given age or a given point in time and (2) the durations of stay in the different stages of life or functional states by members of a cohort or a population. In particular the multistate life table, which summarizes large sets of life history data into experiences of (synthetic) cohorts, has shown to be very useful. Hoem, who was among the first to apply Markov chain models in life contingencies, was also one of the major advocates of the use of probability models in multistate demography and gave the multistate life table a probabilistic underpinning (Hoem and Funck Jensen, 1982). In health sciences (for reviews see Commenges, 1999, and Hougaard, 1999) and labour market studies multistate models are used increasingly to describe and predict entire life histories or biographies. For position papers and illustrations in health sciences, see e.g. Ben-Shlomo and Kuh (2002), Halfon and Hochstein (2002), Peeters et al., (2002) and WHO (2002). For methodological studies integrating multistate demography and health sciences, see Manton and Stallard (1988), Manton et al. (1993), Niessen (2002) and Mamum (2003). The fundamental parameter of multistate models is the (instantaneous) transition rate (continuous-time models) and the transition probability (discrete-time models). The study of how these rates and probabilities change in one or different time scales (e.g. age, calendar time, duration in current position) and what personal and contextual factors affect these changes is the subject of survival analysis, an important subfield of statistics. For a review of statistical models, see e.g. Lancaster (1990), Andersen et al. (1993), Blossfeld and Rohwer (2002), and Hougaard (2002).

The multistate model presented in this paper adds important features to the models covered in the literature. First, the model is an individual model. It describes the financial lifepaths (biographies) of individuals. Individuals are members of a birth cohort and the individual biographies differ from the cohort biographies to the extent that individual differences are taken into account. In this paper, individual differences are random. Hence, members of the same birth cohort have biographies that differ only as a result of chance. Second, the model describes changes in two time scales: individual time (age) and historical or calendar time. As a result, the model is ideally suited to describe changing life histories and to predict life courses. Third, the model brings to insurance and actuarial science key insights from mathematical demography. One example is the relation that exists between individual life cycle behaviour and the aggregate population structure (Preston, 1982; Arthur and Vaupel, 1984). That link is important in translating the commonly held period perspective on financial protection schemes, such as in the PAYGO pension scheme, into a cohort perspective and for disentangling the relative effects of period, cohort and age factors on financial security. Fourth, the model bridges the traditional macro-approach to the study of population and the more recent micro-approach. The model treats changes in population size and composition (macro-level changes) as outcomes of events at the individual level. Fifth, the model is entirely in matrix terms and builds on the matrix methods that have been developed by Rogers (1975, 1995) in multistate demography. As people age they move between different functional states and stages of life. To capture that dynamics, the states must be studied simultaneously. A system of simultaneous linear equations describes that dynamics.

The structure of the paper is as follows. Section 2 reviews the multistate biographic model. The extension to an actuarial model is presented in Section 3. An illustration of the biographic actuarial model to disability insurance is given in section 4. Age-specific transition rates by state of origin and state of destination and payment functions that depend on age and functional state occupied determine the state occupancies, the actuarial values of premiums paid and insurance benefits received and the actuarially fair premium. Section 5 concludes the paper.

2. The multistate model

In a biographic model individuals are characterized by attributes. Sex, marital status, employment status, income status, and health status are some of the attributes. An attribute generally refers to a domain of life, such as family, work and health. One attribute or a combination of attributes defines a functional state and an individual with a given set of attributes is said to occupy a particular state. Not all attributes are manifest. Individuals may differ by latent attributes that remain undisclosed. The biographic model that is considered in this paper is restricted to manifest attributes. Unobserved heterogeneity is disregarded[3]. Attributes vary with age. A change in attribute is an event, which results in a transition from one functional state to a new state. The period between two transitions defines an episode, which is also referred to as a spell and a stage. The duration of an episode, which is equivalent to the duration of stay or sojourn time in a state, is an important characteristic of the live of an individual. The life course may formally be viewed as a sequence of events, states and episodes. If a sequence of events is limited to a particular domain of life, it is often referred to as a career (Elder, 1985; Willekens, 2001). Careers co-exist, co-evolve and interact. Age and year of birth are not treated as attributes. They position the individual in time. Age positions the individual in *individual time* and the year of birth positions the individual in *historical time*. In biographic analysis, populations are stratified in age groups and birth cohorts.

Consider a cohort consisting of m individuals. Individuals are independent and individuals with the same overt characteristics are identical. An individual is denoted by k (k = 1, 2, …, m). At each age and point in time, individual k is characterized by a set of attributes, i.e. occupies a state. The possible states are given by the state space S ={1, 2, 3, …, I}, with I the size of the state space. The state space consists of a finite number of states. If death is considered, the state space includes the state of dead. Dead is an absorbing state; an individual may enter the state but cannot leave the state. The state occupied at a given age or time cannot be predicted with certainty. Hence the indicator variable denoting the state occupied is a random variable. The state individual k occupies at exact age x and exact time t is denoted by the random variable $Y_k(x,t)$. Note that individual k is born at time t-x and is member of the birth cohort t-x. At any time/age, an individual must be in one of I possible states. The random variable $Y_k(x,t)$ is a discrete variable that can take on as many non-zero values as there are states in the state space. It is a polytomous random variable following a multinomial distribution. The sequence of

[3] Hoem (1988, p. 172) point out that actuarial techniques could be developed that build on the notion of unobserved heterogeneity in survival analysis.

random variables $\{Y_k(x,t), x \geq 0 \text{ and } t \geq 0\}$ is a stochastic process identifying the state occupied at each age x and time t. The sequence can be described by a time-inhomogeneous Markov chain with finite state space.

The state occupied may be denoted differently. Let $I_{Y_k(x,t)}$ be an indicator function which is 1 if $Y_k(x,t)$ is i and 0 otherwise, and define ${}_kY_i(x,t) = I_{Y_k(x,t)}$. The expected value of ${}_kY_i(x,t)$ is the probability that individual k is in state i on his x-th birthday, which is precisely at time t: $E\left[{}_kY_i(x,t)\right] = \Pr\left\{{}_kY_i(x,t) = 1\right\}$. It is the *state probability*. Two types of state probabilities are distinguished: unconditional and conditional. The unconditional state probability is the probability that cohort member k occupies state i at age x; it is denoted by ${}_kS_i(x,t)$ with ${}_kS_i(x,t) = E[{}_kY_i(x,t)] = \Pr\{{}_kY_i(x,t)=1\} = \Pr\{Y_k(x,t)=i\}$. The state probabilities are combined in the state vector ${}_k\mathbf{S}(x,t)$ of state probabilities for individual k. It may also be written as ${}_kS_+(x,t) * {}_k\boldsymbol{\pi}(x,t)$ where ${}_kS_+(x,t)$ is the probability of surviving from birth at t-x to exact age x at t irrespective of the state occupied at x and t, and ${}_k\boldsymbol{\pi}(x,t)$ is a vector with elements ${}_k\pi_i(x,t)$ that is the conditional probability that individual k occupies state i at exact age x, provided k is surviving at age x.

The conditional state probability is the probability of occupying state i at age x and time t provided the state occupied at a previous age is known. In addition to the state at a previous age, the personal attributes of individual k at age x, other conditions at age x and attributes and conditions at previous ages may be known. The probability that individual k, who is born at t-x, and who is in state $y_k(x_1,t_1)$, at age x_1 at time t_1, in state $y_k(x_2,t_2)$ at age x_2 at time t_2 and in state $y_k(x_3,t_3)$ at age x_3 at time t_3, and who at age x and time t has attributes $Z_k(x,t)$, is in state j at age x at time t, is

$$Pr\{Y_k(x,t) = j \mid y_k(x_3,t_3), y_k(x_2,t_2), y_k(x_1,t_1); Z_k(x,t)\} \qquad x > x_i \quad i = 1, 2, 3$$

where Z_k may cover characteristics at age x and time t and characteristics and experiences up to age x and time t. In most applications it is assumed that only the most recent state occupancy is relevant:

$$Pr\{Y_k(x,t) = j \mid y_k(x_3,t_3), y_k(x_2,t_2), y_k(x_1,t_1); Z_k(x,t)\} = Pr\{Y_k(x,t) = j \mid y_k(x_3,t_3); Z_k(x,t)\}$$

The covariates $Z_k(x,t)$ are omitted for convenience. If the state occupied at x_3 is i, then

$$\Pr\{Y_k(x,t) = j \mid y_k(x_3,t_3) = i\} = {}_kp_{ij}(x_3,t_3;x,t)$$

where ${}_kp_{ij}(x_3,t_3; x,t)$ is the probability that individual k, who occupies state i at x_3 at time t_3, occupies state j at age x at time t.

The transitions are measured by comparing, for each individual, the states occupied at two consecutive ages or points in time. They are *discrete time transitions* and the probabilities are discrete-time transition probabilities. Transitions may also be measured by recording a movement or transition between two states as it occurs, i.e. in continuous time. These transitions are referred to as *direct transitions*. In multistate and biographic modelling, the distinction between discrete-time transitions and direct transitions is essential. Let ${}_kY_{ij}(x,t)$ be a time-varying indicator variable which takes on the value 1 if individual k whose date of birth is t-x, makes a move from state i to state j at exact age x, i.e. in the infinitesimally small interval following x. It is zero otherwise. The interval is sufficiently small to exclude multiple transitions. During the interval, at most one transition may occur. The expected value of ${}_kY_{ij}(x,t)$ is the probability that individual k born at t-x makes a transition from i to j at exact age x. It depends on being alive at x and being in i at that age. The conditional transition probability is the probability of a move from i to j provided individual k is alive and in state i at age x:

$${}_k\mu_{ij}(x,t) = \lim_{h \to 0} \frac{\Pr\{Y_k(x+h,t+h) = j \mid Y_k(x,t) = i\}}{h} = \lim_{h \to 0} \frac{{}_kp_{ij}(x,x+h,t)}{h} \quad \text{for } i \neq j$$

It is the transition probability per unit time for very small intervals. The probability that individual k who is born at t-x, who is currently of age x and occupies i, moves to j during an infinitesimally small interval following x is known as the instantaneous rate of transition or *transition intensity* at age x and time t, and is denoted by ${}_k\mu_{ij}(x,t)$. It is the conditional probability of a direct transition during an infinitesimally small interval following x.

The unconditional probability of a *direct transition* from i to j at exact age x is the probability of occupying state i at x times the instantaneous rate of transition from i to j. ${}_k\mu_{ij}(x,t)\ {}_kS_i(x,t)$. It may also be written as ${}_kS_+(x,t) * {}_k\pi_i(x,t) * {}_k\mu_{ij}(x,t)$
where ${}_kS_+(x,t)$ is the probability of surviving from birth at t-x to exact age x at t, ${}_k\pi_i(x,t)$ is the probability that a survivor at exact age x occupies i, and ${}_k\mu_{ij}(x,t)$ is the transition intensity at that age.

The multistate model is a continuous time Markov chain. The theory has also been reviewed elsewhere. See e.g. Hoem and Funck Jensen (1982), Namboodiri and Suchindran (1987) and Wolthuis (2003) among others. A continuous time Markov chain (CTMC) is a stochastic process on a discrete state space in continuous time, $\{{}_kY(t);\ t \geq 0\}$, for which the distribution of future states, given the present state and all past states, depends only on the present state and is independent of the past. The CTMC is defined by the instantaneous rates of transition between the states ${}_k\mu_{ij}(x,t)$. The term ${}_k\mu_{ii}(x,t)$ is defined such that $\mu_{ii}(x,t) - \sum_{j \neq i} {}_k\mu_{ij}(x,t) = 0$

Hence

$$-{}_k\mu_{ii}(x,t) = -\sum_{j \neq i}^{I} {}_k\mu_{ij}(x,t) = \lim_{h \to 0} \frac{{}_kp_{ii}(x,x+h,t) - 1}{h}$$

The quantity ${}_k\mu_{ii}(x,t)$ is non-negative.

The matrix of instantaneous rates with off-diagonal elements $-{}_k\mu_{ij}(x,t)$ and with ${}_k\mu_{ii}(x,t)$ on the diagonal is known as the generator of the stochastic process $\{{}_kY(x,t);\ x \geq 0;\ t \geq 0\}$

that describes the position at every age of an individual born at t-x (Çinlar, 1975, p. 256). The matrix is denoted by ${}_k\boldsymbol{\mu}(x,t)$ and has the following configuration:

$$
{}_k\boldsymbol{\mu}(\mathrm{x},\mathrm{t}) = \begin{bmatrix} {}_k\mu_{11}(\mathrm{x},\mathrm{t}) & -{}_k\mu_{21}(\mathrm{x},\mathrm{t}) & . & . & -{}_k\mu_{I1}(\mathrm{x},\mathrm{t}) \\ -{}_k\mu_{12}(\mathrm{x},\mathrm{t}) & {}_k\mu_{22}(\mathrm{x},\mathrm{t}) & . & . & -{}_k\mu_{I2}(\mathrm{x},\mathrm{t}) \\ . & . & . & . & . \\ . & . & . & . & . \\ -{}_k\mu_{1I}(\mathrm{x},\mathrm{t}) & -{}_k\mu_{2I}(\mathrm{x},\mathrm{t}) & . & . & {}_k\mu_{II}(\mathrm{x},\mathrm{t}) \end{bmatrix}
$$

where ${}_k\mu_{ii}(x,t)$ is defined above. Each column of ${}_k\boldsymbol{\mu}(x,t)$ sums to zero.

The probability of a transition during a small interval δx is ${}_k\mathbf{P}(\delta x) = \mathbf{I} + \delta x\ {}_k\boldsymbol{\mu}(x,t)$ (Bartholomew, 1982, p. 86). The transition probability from i to j is ${}_kp_{ij}(\delta x) = \delta x\ {}_k\mu_{ij}(x,t)$ ($i \neq j$). The probability of staying in state i is ${}_k p_{ii}(\delta x) = 1 - \delta x \sum_{j \neq i} {}_k\mu_{ij}(x,t) = 0$. As $\delta x \to 0$, the probability of transition out of any state approaches zero.

The matrix of discrete-time transition probabilities is the transition matrix:

$$
{}_k\mathbf{P}(\mathrm{x}, x+h, \mathrm{t}) = \begin{bmatrix} {}_k p_{11}(x,x+h,t) & {}_k p_{21}(x,x+h,t) & . & . & {}_k p_{I1}(x,x+h,t) \\ {}_k p_{12}(x,x+h,t) & {}_k p_{22}(x,x+h,t) & . & . & {}_k p_{I2}(x,x+h,t) \\ . & . & . & . & . \\ . & . & . & . & . \\ {}_k p_{1I}(x,x+h,t) & {}_k p_{2I}(x,x+h,t) & . & . & {}_k p_{II}(x,x+h,t) \end{bmatrix}
$$

An element ${}_kp_{ij}(x,x+h,t)$ of ${}_k\mathbf{P}(x,x+h,t)$ denotes the (conditional) probability that individual k who is born at t-x and occupies state i at exact age x at time t, is in state j exactly h years later. Each column of ${}_k\mathbf{P}(x,x+h,t)$ sums to one, provided mortality is absent or dead is represented by a separate state.

The dynamics of the multistate system is described by the system of differential equations:

$$
\frac{d\,{}_k\mathbf{P}(x,x+\tau,t)}{d\tau} = -{}_k\boldsymbol{\mu}(x+\tau,t+\tau)\ {}_k\mathbf{P}(x,x+\tau,t)
$$

where ${}_k\boldsymbol{\mu}(x+\tau,t+\tau)$ is the matrix of transition intensities at age x+τ for individuals born at t-x. Note that individual k who occupies state i on his x-th birthday at time t may experience varying transition rates before reaching the age of x+τ at time t+τ.

The differential equation is the forward Chapman-Kolmogorov equation. One element is

$$\frac{d\,{}_kP_{ij}(x,x+\tau,t)}{d\tau} = -{}_k\mu_{j+}(x+\tau,t+\tau)\;{}_kP_{ij}(x,x+\tau,t) + \sum_{r=1}^{I}{}_k\mu_{rj}(x+\tau,t+\tau)\;{}_kP_{ir}(x,x+\tau,t)$$

where ${}_k\mu_{j+}(x+\tau,t+\tau)$ is the intensity of leaving state j. In the forward differential equation, the process is in state i at age x+τ. The probability that the process is in state j changes over the interval [x+τ,x+τ+dτ). The right-hand side is the difference between the outflows from state j during the interval dτ and the inflows into j during the same infinitesimally small interval dτ for a process that is in state i at age x. In demography, the equation is often viewed as a flow equation: the change in (population) stocks is expressed in terms of outflows and inflows.

To solve the system of differential equations, it may be replaced by a system of integral equations:

$${}_k\mathbf{P}(x,x+h,t) = \mathbf{I} - \int_0^h {}_k\boldsymbol{\mu}(x+\tau,t+\tau)\;{}_k\mathbf{P}(x,x+\tau,t)\,d\tau$$

where ${}_k\boldsymbol{\mu}(x+\tau,t+\tau)$ is the matrix of transition intensities at age x+τ for individuals born at t-x and ${}_k\mathbf{P}(x,x+\tau,t)$ is the matrix of transition probabilities during the (x,x+τ)-interval for individuals born at t-x. The integral equation is a flow equation too.

If the transition intensities are constant during the interval from x to x+h (i.e. piecewise constant), the equation may be written as follows:

$${}_k\mathbf{P}(x,x+h,t) = \mathbf{I} - {}_k\boldsymbol{\mu}(x,t)\int_0^h {}_k\mathbf{P}(x,x+\tau)\,d\tau = \mathbf{I} - {}_k\mathbf{M}(x,x+h,t)\;{}_k\mathbf{L}(x,x+h,t) \qquad (1)$$

where ${}_k\mathbf{M}(x,x+h,t)$ is the matrix, with elements ${}_km_{ij}(x,x+h,t)$, of average transition rates during the interval from x to x+h for individuals born at x-t and

${}_k\mathbf{L}(x,x+h,t) = \int_0^h {}_k\mathbf{P}(x,x+\tau,t)\,d\tau$ is the sojourn time spent in each state between ages x and x+h by an individual born at t-x and alive on the x-th birthday, by state at age x. ${}_k\mathbf{L}(x,x+h,t)$ is known as the *exposure function*. The sojourn time in a functional state measures the duration of exposure to the risk of leaving the state. For instance, ${}_kL_{ij}(x,x+h,t)$ denotes the time spent in state j between ages x and x+h by individual k who is born at t-x and who occupies state i at age x.

In the case of piecewise-constant transition intensities, the relation between the transition probabilities and the transition rates is:

$${}_k\mathbf{P}(x,x+h,t) = \exp\left[-h\;{}_k\mathbf{M}(x,x+h,t)\right]$$

which may be approximated by the linear model (see e.g. Willekens, 2006 and the references in that paper)

$${}_k\mathbf{P}(x,x+h,t) = \left[\mathbf{I} + \frac{h}{2}{}_k\mathbf{M}(x,x+h,t)\right]^{-1}\left[\mathbf{I} - \frac{h}{2}{}_k\mathbf{M}(x,x+h,t)\right]$$

The state probabilities at a given age and time depend on the state probabilities at a previous age and the discrete-time transition probabilities during the interval:

$$ {}_k\mathbf{S}(x+h,t+h) = {}_k\mathbf{P}(x,x+h,t)\ {}_k\mathbf{S}(x,t) $$

An element ${}_kS_i(x,t)$ of ${}_k\mathbf{S}(x,t)$ denotes the probability that individual k born at t-x occupies state i at age x at time t.

The previous equation is a projection model. The state probabilities at age x+h and time t+h are related to the state probabilities at a previous age and time.

Note that

$$ \frac{d\,{}_k\mathbf{S}(x+\tau,t+\tau)}{d\tau} = -{}_k\boldsymbol{\mu}(x+\tau,t+\tau)\ {}_k\mathbf{S}(x+\tau,t+\tau)\,d\tau $$

The exposure function ${}_k\mathbf{L}(x,x+h,t)$ is the sojourn time in the different states during the interval from x to x+h by state occupied at exact age x. It is equal to

$$ {}_k\mathbf{L}(x,x+h,t) = \int_0^h {}_k\mathbf{P}(x,x+\tau,t)\,d\tau $$

An element ${}_kL_{ij}(x,x+h,t)$ of ${}_k\mathbf{L}(x,x+h,t)$ is the number of years individual k aged x at exact time t and occupying state i at that time, may expect to spend in state j before reaching age x+h, i.e. during the period t and t+h. An individual who occupies state j at time t is likely to spend more time in j during the (t, t+h)-interval than an individual in another state at t. The elements of ${}_k\mathbf{L}(x,x+h,t)$ are conditional measures since they measure the sojourn time conditional on being in one of the states at time t. The unconditional sojourn time, i.e. the number of years spent in j between t and t+h by an individual born at t-x, irrespective of the state occupied at birth, is

$$ {}_k\mathbf{L}(x,x+h,t)\ {}_k\mathbf{S}(x,t) = {}_k\mathbf{L}(x,x+h,t)\ {}_k\mathbf{P}(0,x,t-x)\ {}_k\mathbf{S}(0,t-x) $$

where ${}_k\mathbf{P}(0,x,t-x)$ is the matrix of discrete-time transition probabilities between ages 0 and x, and ${}_k\mathbf{S}(0,t-x)$ is the vector of state probabilities at birth (at time t-x). An element $p_{ij}(0,x,t-x)$ of ${}_k\mathbf{P}(0,x,t-x)$ is the probability that individual k born in state i at t-x occupies state j at exact age x (and t = x). An element ${}_kS_i(0,t-x)$ of $\mathbf{S}(0,t-x)$ is the probability that child k born at t-x occupies state i at birth.

The sojourn time in j irrespective of the state occupied at exact age x is the vector

$$ {}_{kx}\overline{\mathbf{L}}(x,x+h,t) = {}_k\mathbf{L}(x,x+h,t)\ {}_k\boldsymbol{\pi}(x,t) $$

where ${}_k\boldsymbol{\pi}(x,t)$ is the vector of conditional state probabilities. An element ${}_k\pi_i(x,t)$ is the probability of occupying i at exact age x, conditional on survival at x. An element ${}_{kx}\overline{L}_i(x,t)$ of the vector ${}_{kx}\overline{\mathbf{L}}(x,x+h,t)$ is the sojourn time in i during the interval from x to x+h irrespective of the state occupied at x. It is an unconditional measure of sojourn time.

If the transition intensities are piecewise constant, the exposure function may be expressed in terms of the transition rates (Van Imhoff, 1990; Willekens, 2006):

$$_k\mathbf{L}(x,x+h,t) = \left[_k\mathbf{M}(x,x+h,t)\right]^{-1}\left[\mathbf{I} - \exp\left[-h\ _k\mathbf{M}(x,x+h,t)\right]\right],$$

provided the inverse exists. In demography and actuarial sciences, it is often assumed that events are uniformly distributed during the (x, x+h)-interval. The result is that the exposure function can be approximated by a linear function:

$$_k\mathbf{L}(x,x+h,t) = \frac{h}{2}\left[\mathbf{I} + _k\mathbf{P}(x,x+h,t)\right]$$

Now we turn to direct transitions. If individual k makes a direct transition from i to j at time t and age x, $_kY_{ij}(x,t) = 1$. It is zero otherwise. The probability that individual k, born at t-x and currently aged x and occupying state i, makes a direct transition to state j is $E[_kY_{ij}(x)] = {_k\mu_{ij}(x,t)}\ {_kS_i(x,t)}$. Let $_kN_{ij}(x,t)$ denote the number of direct transitions individual k makes during the interval (x, x+dx; t, t+dt). It is the density of (i,j)-transitions at age x and time t. The sequence $\{_kN_{ij}(x,t);\ x \geq 0 \text{ and } t \geq 0\}$ is a stochastic process known as a multivariate counting process. The theory of counting processes was first developed by Aalen (1975, 1978) in his PhD dissertation. For a complete overview, see Andersen et al. (1993) and for a brief introduction Hosmer and Lemeshow (1999, Appendix 2). During the infinitesimally small interval, at most one transition is possible, hence $_kN_{ij}(x,t)$ is 0 or 1. The expected number of (i,j) transitions during the interval is

$$E\left[_kN_{ij}(x,t)\right] = {_k\mu_{ij}(x,t)}\ {_kS_i(x,t)}\,dx$$

The number of direct (i,j)-transitions individual k, who is born at t-x, may expect to make during the age interval from x to x+h, is

$$E\left[_{k0}N_{ij}(x,x+h,t)\right] = \int_0^h {_k\mu_{ij}(x+\tau,t+\tau)}\ {_kS_i(x+\tau,t+\tau)}\,d\tau$$

It is an unconditional measure, i.e. it depends on the probability of surviving and being in state i at age x+τ. The count measure is determined by an observer at the reference age 0. The expected number of direct transitions during the interval may also be viewed from a different reference age. The expected number of direct (i,j)-transitions individual k, who is born at t-x *and alive at age x*, may expect to experience during the interval (x,x+h) is

$$E\left[_{kx}N_{ij}(x,x+h,t)\right] = \int_0^h {_k\mu_{ij}(x+\tau,t+\tau)}\frac{_kS_i(x+\tau,t+\tau)}{_kS_+(x,t)}\,d\tau$$

where x is the reference age and $_kS_+(x,t)$ is the probability that individual k is alive at age x at time t. The state occupied at that age is not relevant.

The expected number of direct transitions beyond age x is

$$E\left[_{kx}N_{ij}(x,\infty,t)\right] = \int_0^\infty {_k\mu_{ij}(x+\tau,t+\tau)}\frac{_kS_i(x+\tau,t+\tau)}{_kS_+(x,t)}\,d\tau$$

Let ${}_{kx}\mathbf{N}(x,x+h,t)$ denote the matrix of expected numbers of direct transitions during the interval from x to x+h by individual k who is exact age x at t. It is

$$ {}_{kx}\mathbf{N}(x,x+h,t) = \frac{1}{{}_{k}S_{+}(x,t)} \int_0^h {}_{k}\boldsymbol{\mu}(x+\tau,t+\tau)\ {}_{kd}\mathbf{S}(x+\tau,t+\tau)\, d\tau $$

where ${}_{kd}\mathbf{S}(x,t)$ is a diagonal matrix with the state probabilities ${}_{k}S_{i}(x,t)$ in the diagonal. The unconditional measure is the number of (i,j)-transitions in the interval from x to x+h by a new-born aged 0. The number of direct (i,j)-transitions between x and x+h by state at age y < x is $\int_0^h {}_{k}\boldsymbol{\mu}(x,t)\ {}_{k}\mathbf{P}(x,x+\tau,t)\, d\tau$

If transition intensities are constant during the (x, x+h)-interval, the product ${}_{k}\mathbf{M}(x,x+h,t)\ {}_{kx}\overline{\mathbf{L}}(x,t)$ gives the expected numbers of direct transitions experienced during the age interval (x,x+h) by individual k aged x at time t. Note that the probability of a discrete-time transition during the same interval by individual k aged x at t is ${}_{k}\mathbf{P}(x,x+h,t)\ {}_{k}\boldsymbol{\pi}(x,t)$.

The biographic model has been developed for individual k. It may easily be extended to a cohort model. Let Q denote the number of children born in a given year denoted by t-x-1. It covers the period from t-x-1 to t-x. These children reach their x-th birthday during the year from t-1 to t. At time t they are x years old in completed years, i.e. they are between x and x+1 years of age. The state child k occupies at birth is denoted by $Y_k(0,T-x)$ where T varies from t-1 to t. The distribution of newly born children between the functional states given by the vector of state probabilities $\mathbf{S}(0,t-x-1)$ where t-x-1 denotes the year of birth. The distribution between functional states of individuals celebrating their x-th birthday is

$$ \mathbf{k}(x,t-1) = \mathbf{P}(0,x,t-x-1)\mathbf{S}(0,t-x-1)Q $$

3. The actuarial model

Each state occupancy and state transition may involve a payment. For each state occupancy and state transition a *payment function* indicates what needs to be paid and who should pay whom. By associating payments to state occupancies and state transition, a biographic actuarial model emerges that is generic and encompasses most if not all insurance schemes that exist today. In private insurance, the payment function is specified by an insurance contract or insurance policy. In social insurance and social security payments are based on collective agreements (entitlements) and have a legal basis. The first part of this section describes several factors that cause insurance policies to differ. These factors are embedded in payment functions that represent a major component of the actuarial model that is presented next. The actuarial model encompasses a wide variety of insurance contracts covering a range of contingencies in the life course. A contract involves two parties, the insured or beneficiary and the insurer. The insurer can be private or a public body and the contract can be a real contract, a social contract agreed upon by a collective or a legal arrangement. In this paper, *insurer* is used to indicate the actor who expresses the commitment to pay the beneficiary. It includes the private and the public sector, insurance companies, pension funds, and any organization that agrees to pay individuals during particular episodes of life and/or transitions in life and receives compensation in return.

The insurance contract describes the actors (insured, insurer), the period (term) of the insurance, the coverage, the premium, the benefit or claim and other conditions. The social contract governing a collective insurance specifies who is eligible to participate in the insurance and what the conditions are. The period of insurance (policy period) extends from the onset of the insurance contract (policy issue) to the end of the contract (term of policy). During that period the insured pays premiums to the insurer and receives benefits in return. The period of the insurance can be fixed or variable. In case of a fixed period of length n, say, the insurance is a *term insurance* of duration n. Coverage may start at the date the policy is issued, the date at which an event occurs (e.g. onset of disability or death) or after a waiting period (e.g. deferred period, deferred annuity).

Payments may be lump sum payments at one point in time, a series of payments at predetermined points in time or a continuous payment. Lump sum payments are usually linked to state transitions and to state occupancies at a predetermined point in time. Continuous payments are linked to episodes, i.e. periods of state occupancies. The insured or policyholder pays a premium to maintain a claim on the insurance contract. The premium may be paid as a single payment (lump sum; *single premium*) at one moment in time, e.g. at the time the insurance policy is issued, or as monthly or annual payments during a predefined period of time. The insurance benefit paid to the insured by the insurer may also be a single payment or monthly or annual payments. One example of a single benefit payment is the *endowment assurance*, in which a given amount is paid in case of death or survival to maturity (date on which payment is due). If the payment is made only if death occurs within a period of n years, the insurance is a term insurance. If the payment is made at time of death irrespective of when it occurs, the insurance is known as a *whole life insurance*. Another example of a single benefit payment is an insurance contract providing a lump sum benefit in case of permanent disability. The payment of a lump sum may also be at a point in time that is unrelated to the occurrence of a life event but that is related to either the onset of the insurance contract (*policy issue*) or the end of the contract (*term of the policy*). Benefits that are paid at equal intervals are known as *annuities*. The annuity owner or *annuitant* is the person entitled to receive annuity payments. The payment is usually made monthly or annually and the period can be fixed or variable. An *annuity certain* is annuity payment during a predetermined number of years, regardless of life or death. Variable periods arise when the payment is associated with a state occupancy and is made as long as the policyholder or beneficiary occupies a given state. Examples of variable periods include the remaining lifetime (in case of life insurance), the duration of employment and unemployment, the duration of disability. Hence, the state occupied determines how much is paid and the sojourn time in a state determines the duration of payment. A *life annuity* consists of a series of payments that are made as long as the insured is alive. It provides an income for life. A *disability annuity* is a series of payments made while the insured is disabled. A combination of annuities is possible, for instance an annuity certain for 10 years, say, and a deferred annuity beyond that. Keyfitz and Rogers (1982, p. 67) consider such an annuity policy. The level of payment may be fixed or variable. For instance, a *level premium* is a premium that does not change for the entire duration of the policy.

Payments may include administrative expenses made by the insurer in the operation of an insurance contract, a savings/investment component, and taxes. The fees or administrative expenses included in the premium are known as *load*. A premium that covers the insured benefits and the incurred costs is known as the expense-loaded premium (or adequate premium) (see e.g. Berger, 1995, pp. 104ff). A payment often includes an investment or a return on investment in addition to insurance. In that case, the annuity is known as variable annuity because the benefit payment is linked to the values of investments, such as common stocks. In the United States, variable annuities account for approximately two-thirds of the annuity sales (Brown and Poterba, 2005). A fixed annuity guaranties a fixed amount monthly. Analogously, variable life insurance has an investment component and the amount of benefit paid depends on the value of the assets behind the contract. Taxes play a significant role in insurance. Some premiums are taxed while other premiums are not. Of particular interest are tax deferral schemes in which premiums paid are tax-exempt while benefits received are taxed. *Universal life* is a life insurance that includes, in addition to insurance, a savings component that is invested in a tax-deferred account. It allows the holder to shift money between the insurance and savings components of the policy. In the early 1980s (1983), the Allstate Life Insurance Company introduced Universal Life as a flexible life insurance policy and marketed the product as *One policy for a lifetime of changing needs*. Today, universal life is part of life planning which is a holistic approach to financial challenges at all stages of life and combines different financial threats into a single scheme. The biographic actuarial model is a technical instrument to implement Universal Life and other holistic financial protection plans that cover the entire life course.

The transfer of risk from the insured to the insurer involves the payment of a premium to compensate the one who takes over the risk. The compensation is determined on the basis of the premium principle that assigns to the risk a real number. That number is used as the financial compensation for the risk transfer. It depends on the risk transferred, administrative and other costs incurred by the insurer, and a loading to compensate the insurer for being in a less safe position and to avoid getting in ruin. The loading depends on the degree of risk aversion of the insurer. The loading may also include “adverse selection” costs associated with voluntary purchase behaviour. Adverse selection exists when the survival or transition probabilities of people who purchase insurance or a financial protection plan (e.g. pension plan) are different from those of the general population. Poterba and Warshawsky (2000) estimate that for commercial insurers in the United States offering life annuity payouts purchases with funds from the individual accounts, the present value of the benefits, using the mortality rates of the general population, is between 15 to 25 percent below the present value of the premium payment. Premium principles differ in the factors they consider and the way the factors are treated. Kaas et al. (2001, pp. 113ff) discuss several premium principles. In this paper, we consider only risk premiums and disregard administrative expenses made by the insurer and other surcharges that may apply. The premium is determined from the distribution of the claims or insurance benefits. The minimum premium is sufficient to cover the insurance benefits. If the insurer is risk neutral, the loading for risk aversion is zero. In this case the premium principle is the *equivalence principle*. Following that principle, the premium is determined by equating at policy issue the expected present value of future premiums to the expected present value of insurance benefits or claims. The premium that satisfies the equivalence principle is a *net premium*. The net premium is sufficient for a risk neutral insurer. In this paper, we assume a risk neutral insurer. The expected present value of a payment or a series of payments is the *actuarial value* of the payment(s). The actuarial value depends on a set of actuarial assumptions embedded in the premium principle and on the timing of payments, which in turn depends on the timing of transitions in the life course. The likelihood of transitions, the timing of transitions and the sojourn times or durations of stay in several functional states can be predicted by the biographic model.

The actuarial model presented in this paper is generic in that it considers several types of payments. The types belong to two broad classes: lump sum payments and continuous payments. Continuous payments are governed by intensities or instantaneous rates of payment, as suggested by Hoem (1988, p. 174), Wolthuis (2003, p. 3) and others. The following types of payments are distinguished (Haberman and Pitacco; 1999, p. 3):

a. A continuous contribution or premium paid during the infinitesimal interval (x, x+dx) by individual k (the insured or policy holder) who is in state i at age x and time t[4]. It is the *instantaneous premium rate* ${}_kp_i(x,t)$. The amount paid during the interval (x, x+dx) is ${}_kp_i(x,t)\,dx$. The premium paid at age x depends on the state occupied at that age. An individual who is employed may pay a different premium than an individual who is unemployed, retired or disabled.
b. A continuous annuity benefit paid by the insurer during the infinitesimal interval (x, x+dx) to individual k who is in state j at age x and time t. It is the *instantaneous benefit rate* ${}_kb_j(x,t)$. The amount the beneficiary receives during the interval (x, x+dx) is ${}_kb_i(x,t)\,dx$. The benefit may vary with the state occupied. For instance, a severely disabled individual may receive a higher benefit than a mildly disabled individual.
c. A lump sum ${}_kc_{ij}(x,t)$ paid by the insurer at time t to individual k if, at that point in time, individual k experiences a (direct) transition from state i to state j. The lump sum is associated with the (i,j)-transition. If the transition does not occur, the lump sum is not paid. Hence the probability of receiving a benefit is the probability of a transition at time t. At time t individual k is aged x. The payment of the benefit may be restricted to transitions that occur within a given period (t, t+n) or age interval (x, x+n).
d. A lump sum ${}_kd_j(x,t)$ paid by the insurer at time t to individual k if, at that point in time, individual k is in state j. The lump sum is associated with the state occupancy at t. If the insured k is not in j at t, the lump sum is not paid. The time t is generally the point in time when the policy expires. The lump sum is known as *pure endowment*. The probability of receiving a benefit is the state probability at

[4] In actuarial science, it is said that the policy is in state i.

age x and time t. In life insurance, a pure endowment provides for payment of the sum insured only if the individual insured is alive at t.

e. A lump sum $_kg(x,t)$ paid at time t by an insured k to the insurer. An example is the level premium.

Special cases of the above cases are considered by Wolthuis (2003, pp. 26ff). They include insurance benefits that are functions of premium reserves at time t, where premium reserves may cover all premiums paid previously or may be restricted to premiums paid while in a given state.

Suppose that at time t an individual k of age x pays a premium if he is in state i and receives an annuity benefit if he is in state j. In the following section, t is omitted for convenience. The benefit received during the infinitesimal interval [x, x+dx) is denoted by $_kb_j(x)$. The benefit received at age x depends on the state occupied at that age. Since the state occupied at x cannot be predicted with certainty it is represented by a random variable, $_kY_j(x)$ say. As a consequence, the benefit received is a random variable too. It is equal to $_kb_j(x)\ _kY_j(x)\,dx$. If $_kY_j(x) = 1$, individual k receives the benefit, otherwise not. The present value (PV) at birth of that benefit is $\exp(-\delta x)\ _kb_j(x)$, where δ is the instantaneous rate of interest, which is assumed to be non-stochastic and constant. The expression $\exp(-\delta x)$ is the discount function and $\exp(-\delta)$ the annual discount factor. The PV of the benefit individual k receives at age x is

$$_k\mathrm{B}_j(x) = \exp[-\delta\, x]\ _kb_j(x)\ _kY_j(x)\,dx$$

The present value is a random variable (see also Haberman and Pitacco, 1999, p. 48)

The random present value of the annuity benefit received during the age interval from x to x+h is

$$_k\mathrm{B}_j(x,x+h) = \int_x^{x+h} \exp[-\delta\xi]\ _kb_j(\xi)\ _kY_j(\xi)\,d\xi$$

The present value may be estimated at reference ages different from age 0 (birth). The present value at age x is

$$\int_x^{x+h} \exp[-\delta(\xi - x)]\ _kb_j(\xi)\ _kY_j(\xi)\,d\xi$$

The *expected* PV of the benefit received at age x by individual k is

$$E[_k\mathrm{B}_+(x)] = \sum_{j=1}^{I} \exp[-\delta\, x]\ _kb_j(x)\,\mathrm{E}[_kY_j(x)]$$

where $E[_kY_j(x)]$ is the expected value of $_kY_j(x)$. It is equal to the state probability $_kS_j(x) = \Pr\{_kY_j(x)=1\} = \Pr\{Y_k(x)=j\}$, which is the probability that individual k is in state j at exact age x. The actuarial value of an annuity benefit paid by the insurer during the period from x to x+h provided the insured is in state j is

$$E[_k\mathrm{B}_j(x,x+h)] = \int_x^{x+h} \exp[-\delta\xi]\ _kb_j(\xi)\ _kS_j(\xi)\,d\xi$$

If the beneficiary k occupies state j only during part of the (x,x+h)-interval, the annuity is paid only during the sojourn time in j.

If the annuity level is one (unit-level annuity), the actuarial value is[5]

$$E\left[{}_k\mathrm{B}_j(x,x+h)\right]=\int_x^{x+h}\exp\left[-\delta\xi\right]{}_kS_j(\xi)\,d\xi$$

If annuities are paid to people in different states, the annuity benefit depends on duration of stay in the different states. The expected present value is

$$E\left[{}_k\mathrm{B}(x,x+h)\right]=\int_x^{x+h}\exp\left[-\delta\xi\right]\sum_{j=1}^{I}\left[{}_kb_j(\xi)\ {}_kS_j(\xi)\right]d\xi$$

Different states could relate to different degrees of severity of disability, or to unemployment and disability. The above expression provides a basis for a comprehensive insurance policy that includes several domains of life. Note that the states should be mutually exclusive.

The premium paid while in state i during the infinitesimal interval [x, x+dx) is ${}_kp_i(x)$. The PV of the premium paid is determined in a way analogous to the benefit. Let ${}_k\Pi_i(x,x+h)$ denoted the PV of the premium paid during the interval from x to x+h when individual k is in state i. It is obtained by the following expression:

$${}_k\Pi_i(x,x+h)=\int_x^{x+h}\exp\left[-\delta\xi\right]{}_kp_i(\xi)\ {}_kY_i(\xi)\,d\xi$$

The actuarial value of the premium paid is the expected value

$$E\left[{}_k\Pi_i(x,x+h)\right]=\int_x^{x+h}\exp\left[-\delta\xi\right]{}_kp_i(\xi)\ {}_kS_i(\xi)\,d\xi$$

If the premium differs by state occupied, then the PV of the premium paid by individual k at age x is

$$E\left[{}_k\Pi(x,x+h)\right]=\int_x^{x+h}\exp\left[-\delta\xi\right]\sum_{i=1}^{I}\left[{}_k\,p_i(\xi)\ {}_kS_i(\xi)\right]d\xi$$

The present value of a lump sum paid by the insurer to individual k who experiences a transition from i to j is determined in a similar way. Assume that the transition occurs at time t when individual k is aged x. The individual is born at t-x. The value of the lump sum associated with the (i,j)-transition is ${}_kc_{ij}(x,t)$. The lump sum is paid if individual k born at t-x makes an (i,j)- transition at t. Recall that ${}_kY_{ij}(x,t)$ is a time-varying indicator variable which takes on the value 1 if individual k whose date of birth is t-x, makes a move from state i to state j at exact age x, i.e. in the infinitesimally small interval following x. It is zero otherwise. The random present value of the lump sum, measured at birth of the insured, is

$${}_k\mathrm{B}_{ij}(x)=\exp\left[-\delta\,x\right]{}_kc_{ij}(x,t)\ {}_kY_{ij}(x,t)$$

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[5] In the actuarial literature, the notation $\bar{a}^{j}_{x:h}$ is used to denote the expected present value, measured at time 0, of a unit-level annuity benefit paid during the period from x to x+h if the insured is in state j.

The *expected* present value of the lump sum paid at time of the transition during the infinitesimal interval (x, x+dx) is

$$E\left[{}_k\mathrm{B}_{ij}(x)\right] = \exp\left[-\delta\, x\right]{}_k c_{ij}(x,t)\ {}_k\mu_{ij}(x,t)\ {}_k S_{ij}(x,t)\,dx$$

where ${}_k\mu_{ij}(x,t)\ {}_kS_i(x,t)$ dx is the probability density of a transition during the interval (x, x+dx) and ${}_kc_{ij}(x,t)$ is the amount individual k receives provided the transition takes place. The probability density may be denoted by ${}_kf_{ij}(x,t)$: ${}_kf_{ij}(x,t) = {}_k\mu_{ij}(x,t)\ {}_kS_i(x,t)$.

If a lump sum is paid at each (i,j)-transition experienced during an interval from x to x+h and the amount depends on age, then the expected present value of the lump sums is

$$E\left[{}_k\mathrm{B}_{ij}(x,x+h)\right] = \int_x^{x+h}\exp\left[-\delta\,\xi\right]{}_k c_{ij}(\xi)\ {}_k\mu_{ij}(\xi)\ {}_k S_i(\xi)\,d\xi$$

If the lump sum is independent of age or time at transition, then

$$E\left[{}_k\mathrm{B}_{ij}(x,x+h)\right] = {}_k c_{ij}\int_x^{x+h}\exp\left[-\delta\,\xi\right]{}_k\mu_{ij}(\xi)\ {}_k S_i(\xi)\,d\xi = {}_k c_{ij}\ E\left[{}_{kx}ND_{ij}(x,x+h)\right]$$

where $E\left[{}_{kx}ND_{ij}(x,\infty)\right]$ is the expected *discounted* number of direct transitions from state i to state j between ages x and x+h by individual k aged x at t[6].

If a lump sum is paid for every (i,j)-transition beyond age x, then the expected value of the benefits received beyond age x discounted to age x is

$$E\left[{}_{kx}\mathrm{B}_{ij}(x,\infty)\right] = \int_x^{\infty}\exp\left[-\delta(\xi - x)\right]{}_k c_{ij}(\xi)\ {}_k\mu_{ij}(\xi)\,\frac{{}_k S_i(\xi)}{{}_k S_+(x)}\,d\xi$$

where ${}_kS_+(x)$ is the probability that individual k is alive at exact age x. The state occupied at that age is not relevant. If the lump sum is independent of age, then

$$E\left[{}_{kx}\mathrm{B}_{ij}(x,\infty)\right] = {}_k c_{ij}\ E\left[{}_{kx}ND_{ij}(x,\infty)\right]$$

The final type of benefit payment is one where the insurer pays a lump sum benefit to individual k if he is in state j at age x (and time t). The lump sum is $d_j(x)$. The expected present value of a lump sum paid by the insurer at time t to the individual k is $\exp[-\delta x]\ d_j(x)\ {}_kS_j(x)$. It is the product of the probability of being in state j at age x (at time t) and the discounted value of the lump sum benefit.

The actuarial value at age x of all benefits provided by an insurance policy (annuity benefits and lump sum payments) during the interval from x to x+h is (see Haberman and Pitacco, 1999, p. 52)

[6] The concept of discounted number of events is introduced by Fisher in the field of genetics when he developed the theory of reproductive value (for a recent review, see Keyfitz and Caswell, 2005). It is consistent with the concept of 'discounted number of survivors' used in actuarial sciences (see e.g. Gerber, 1995, p. 120).

$$E\left[{}_k\mathrm{B}(x,x+h)\right] = \frac{1}{S_+(x)}\left[\int_x^{x+h} \exp\left[-\delta(\xi - x)\right]\sum_{j=1}^{I}\left[b_j(\xi)\ {}_kS_j(\xi)\right]d\xi\right.$$
$$+\int_x^{x+h} \exp\left[-\delta(\xi - x)\right]\sum_{j=1}^{I}\sum_{k=1}^{I}\left[c_{jk}(\xi)\ {}_k\mu_{jk}(\xi)\ {}_kS_j(\xi)\right]d\xi$$

$$\left.+\int_x^{x+h} \exp\left[-\delta(\xi - x)\right]\sum_{j=1}^{I}\left[d_j(\xi)\ {}_kS_j(\xi)\right]d\xi\right]$$

where the benefit is for an individual alive at age x, without consideration of the state occupied at x. ${}_kS_j(\xi)$ is the probability that individual k is alive at age ξ and occupies state j at age ξ. It is equal to the probability of being alive at ξ, ${}_kS_+(\xi)$, times the conditional probability of occupying state j, ${}_k\pi_j(\xi)$. Note that the actuarial value of the insured benefit is expressed in terms of the state probabilities at ξ. The benefit at ξ may also be estimated conditional on the state occupied at age x (x ≤ ξ), in which case the actuarial value is expressed in terms of discrete-time transition probabilities.

Individual k pays a premium when in one of several states and receives a benefit when in one of some other states. The difference between premiums paid and benefits received varies over the life course. The insurer has a premium reserve if the actuarial value of future insurance benefits exceeds the actuarial value of future premiums. The insurer holds the reserve for fulfilment of the policy obligations. The reserve may be defined prospectively or retrospectively. The *prospective premium reserve* for an insurance policy at age x is the actuarial value of the future benefits less the actuarial value of the future premiums: $E\left[{}_k\mathrm{B}(x,x+h)\right]-E\left[{}_k\Pi(x,x+h)\right]$. It is a summary of benefit minus premium payment streams between x and x+h. The prospective reserve for all policies is $\sum_{k=1}^{m}\left[E\left[{}_k\mathrm{B}(x,x+h)\right]-E\left[{}_k\Pi(x,x+h)\right]\right]$ (Haberman and Pitacco, 1999, p. 53; Wolthuis, 2003, p. 31). The *retrospective premium reserve* of an insurance policy is the actuarial accumulated value of the past premiums minus past benefits. It is a summary of the premium minus benefit payment streams leading up to x. The retrospective premium reserve was introduced by Hoem (1988). For details, see Wolthuis (2003, pp. 186ff). The retrospective reserve at age x is defined over the interval (0, x) while the prospective reserve is defined over the interval (x, ∞). If the prospective reserve is positive, the expected benefits that need to be paid to policyholders in the future exceed the expected premiums to be collected. In the absence of a retrospective reserve, the insurer has a funding requirement. To restore the balance, premiums may be raised or benefits may be reduced[7]. The prospective and the retrospective reserve may be defined for each state in the state space (Wolthuis and Hoem, 1990).

The *equivalence principl*e is fulfilled if at policy issue the expected value of future benefits is equal to the expected value of future premiums. More formally, the actuarial

[7] For instance in Dutch occupational pension plans, pensions are adjusted for inflation. The adjustment is not complete implying a decline of purchasing power of pensioners. The rate of adjustment may depend on the prospective reserve, as is the case with the largest pension fund, the ABP fund covering civil servants.

value of the premiums paid between the policy issue (age 0) and the term of the insurance (age ω) is equal to the actuarial value of the benefits:

$$E\left[{}_k\mathrm{B}(0,\omega)\right] = E\left[{}_k\Pi(0,\omega)\right]$$

If the actuarial value of the stream of benefit payments exceeds the actuarial value of the stream of premiums, the insurer incurs a loss. To prevent a loss, a premium is determined that satisfies the equivalence principle. That premium is the *net premium*. The premium reserve considered in this section is net, since it is assumed that the insurer is risk neutral and no expenses are included in the benefit and premium functions. If the premium individual k pays is determined by the equivalence principle, the premiums paid cover the expected benefit payments. The insurance scheme is *actuarially fair* at the individual level. Actuarial fairness is usually determined at the group level. If for a group of m individuals the net premium is determined such that $\sum_{k=1}^{m} E\left[{}_k\mathrm{B}(0,\omega)\right] = \sum_{k=1}^{m} E\left[{}_k\Pi(0,\omega)\right]$, then the insurance organization can meet its obligations (net of administrative expenses) and the insurance system is actuarially fair for the group although some members of the group may benefit more than others (for a discussion of distributional effects in actuarially fair insurance schemes, see Caselli et al., 2003). If the equivalence principle is the basis for setting the premium, the insurance scheme is actuarially fair.

4. Illustration: disability insurance

To illustrate the actuarial model, we consider an example in the field of disability insurance. Suppose that individual k purchases a disability insurance on his x-th birthday. Assume that the insurance contract stipulates that the insurer pays a constant disability annuity b to individual k during periods of disability that occur before k reaches age ω. Beyond age ω the insurance does not cover anything. The term of the insurance policy is ω - x. In this example, the disability benefit paid by the insurer does not depend on the degree of disability nor on age. The premium is assumed to vary with age. The premium is paid when individual k is active (not disabled) and is interrupted during periods of disability. The premium function is denoted by p(x). Consider a three-state model (Figure 1). The states are active, disabled and dead. The transition intensities are shown in Figure 1. The intensity ${}_k\mu_{12}(x)$ is the instantaneous rate of entry into disability at age x. In the epidemiological literature, it is known as the instantaneous incidence rate. In the actuarial literature it is the rate of inception of disability. ${}_k\mu_{21}(x)$ is the instantaneous rate of recovery (exit from disability) at age x. The force of mortality at age x is ${}_k\mu_{13}(x)$ if individual k is active and ${}_k\mu_{23}(x)$ if he is disabled.

Figure 1 about here

Suppose individual k is healthy and active at age x when the policy is issued. The changes in state probabilities are described by the system of differential equations

$$\frac{d\,{}_k\mathbf{P}(x, x+\tau)}{d\tau} = -{}_k\boldsymbol{\mu}(x+\tau)\;{}_k\mathbf{P}(x, x+\tau)$$

where an element ${}_kp_{ij}(x,x+\tau)$ denotes the probability that individual k who is in state i at exact age x (e.g. age at policy issue) is in state j at age x+τ.

The solution of the system is

$${}_k\mathbf{P}(x, x+h) = \exp\left[-\int_0^h {}_k\boldsymbol{\mu}(x+\tau)\,d\tau\right]$$

If the transition intensities are constant during the interval from x to x+h, the matrix of transition probabilities is

$${}_k\mathbf{P}(x, x+h) = \exp\left[-{}_k\mathbf{M}(x)\int_0^h d\tau\right] = \exp\left[-h\;{}_k\mathbf{M}(x)\right]$$

where ${}_k\mathbf{M}(x) = {}_x\boldsymbol{\mu}(x+\tau)$ for $0 \le \tau < h$.

The state probabilities at age x+h are given by the equation:

$${}_k\mathbf{S}(x+h) = {}_k\mathbf{P}(x, x+h)\;{}_k\mathbf{S}(x) = \exp\left[-h\;{}_k\mathbf{M}(x)\right]{}_k\mathbf{S}(x)$$

The sojourn times are given by the exposure function

$${}_k\mathbf{L}(x, x+h) = \int_0^h {}_k\mathbf{P}(x, x+\tau)\,d\tau$$

At age x, individual k is in state 1. The premium he may expect to pay at age x+τ and the annuity benefit he may expect to receive depend on the state occupied at age x+τ. If he is in state 1, he pays an age-specific premium p(x+τ) and if he is in state 2, he receives a fixed benefit b. The following matrix expression produces the expected present values at age x of the premium paid and benefit received at age x+τ

$$\exp[-\delta\tau]\begin{bmatrix} -p(x+\tau) & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} {}_kP_{11}(x, x+\tau) & 0 \\ {}_kP_{12}(x, x+\tau) & 0 \end{bmatrix}$$

where ${}_kP_{1j}(x,x+\tau)$ is the probability that individual k, who is in state 1 at age x (i.e. at time the insurance policy is issued), is in state j at age x+τ (j = 1, 2). Note that ${}_kP_{11}(x,x+\tau)$ + ${}_kP_{12}(x,x+\tau)$ may be less than one because of mortality. The premium paid receives a minus sign because it involves a cost to individual k. In the actuarial literature, the premium is positive and the benefit is negative (see e.g. Hoem, 1988, p. 196; Ramlau-Hansen, 1988, p. 226). The prospective gain from the disability insurance over the entire lifetime of individual k or the term of the insurance, whatever comes first, is

$$\begin{bmatrix} {}_kGain(x,\omega) & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}\int_0^{\omega - x}\left[\exp[-\delta\tau]\begin{bmatrix} -p(x+\tau) & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} {}_kP_{11}(x, x+\tau) & 0 \\ {}_kP_{12}(x, x+\tau) & 0 \end{bmatrix}d\tau\right]$$

The actuarial value of the insured benefit is expressed in terms of the probability of being disabled at age x+τ [= ${}_kP_{12}(x,x+\tau)$] and the actuarial value of the premium paid is expressed in terms of the probability of being active [= ${}_kP_{11}(x,x+\tau)$].

A numerical example will clarify the actuarial model further. We consider the example given by Haberman and Pitacco (1999, p. 58ff) but we reformulate the example using matrices. Suppose an individual of age 40 purchases a disability insurance. The term of the insurance is 10 years, i.e. if the policyholder becomes disabled before age 50, the insurer pays a constant annuity of one (b = 1) until age 50. The insurance considered is therefore a *term disability insurance*. The disability insurance starts paying immediately at onset of disability and does not enforce a waiting period that is common in disability insurances. A constant premium is paid for 5 years while the policyholder is active. No premium is paid during episodes of disability. The instantaneous rate of interest is ln(1.04) which is 3.9221 percent. The transition intensities are assumed constant. They are $\mu_{12} = 0.002136$, $\mu_{13} = 0.004183$, $\mu_{21} = 0.005$, $\mu_{23} = 1.2*\mu_{13} = 0.005020$. The factor 1.2 is the excess mortality associated with disability. The instantaneous rate of death is 20 percent higher for disabled individuals than for active individuals. It is a relative risk.

The state probabilities are

$$\mathbf{S}(x+1) = \exp[-\mathbf{M}]\mathbf{S}(x) \approx [\mathbf{I} + \tfrac{1}{2}\mathbf{M}]^{-1}[\mathbf{I} - \tfrac{1}{2}\mathbf{M}]\mathbf{S}(x)$$

where **M** is constant during the age interval from 40 to 50:

$$\mathbf{M} = \begin{bmatrix} 0.002136 + 0.004183 & -0.005 \\ -0.002136 & 0.005 + 0.005020 \end{bmatrix} = \begin{bmatrix} 0.006319 & -0.005 \\ -0.002136 & 0.01002 \end{bmatrix}$$

The matrix of transition probabilities is determined using the linear approximation. It is

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{21} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} 0.9937 & 0.0050 \\ 0.0021 & 0.9900 \end{bmatrix}$$

Table 1 and Figure 2 show how the state probabilities evolve with duration of the policy. At entry into the disability insurance, policyholders are active. At age 50, i.e. at the policy term, about 4 percent of the policyholders is dead and 2 percent is disabled. Of those who die before age 50, 99 percent dies while active and 1 percent dies while disabled. The share of deaths of disabled individuals among the total deaths depends on the incidence of disability **and** the mortality level among disabled individuals.

Table 1 about here
Figure 2 about here

Table 2 shows the sojourn times in each state by age. The sojourn times are estimated assuming uniform distribution of transitions within age intervals of one year. The assumption is equivalent to the assumption of piecewise constant probability densities. The expected sojourn time in the active state during the policy, given the transition rates, is 9.69 years; the duration of disability is 0.10 year, i.e. a little over a month.

Table 2 about here

When an individual is disabled, he receives a disability annuity benefit of one unit (b = 1). The nominal value of the benefit received is b $_{40}\overline{L}_2(40,50) = 0.1011$ units. The expected value of the total benefit received between ages 40 and 50, discounted at age 40, is

$$E[\mathrm{B}(40,50)] = b\sum_{t=0}^{9} \exp[-0.039221(t+0.5)]\ _{40}\overline{L}_2(40+t, 40+t+1)$$

where $_{40}\overline{L}_2(40+t, 40+t+1)$ is the sojourn time in the disability state between age 40+t and 40+t+1 and b is the annuity benefit (equal to one). The actuarial value of the insured benefit is B(40,50) = 0.0784. The value depends on the annuity benefit, the interest rate, the incidence of disability, the recovery rate and the death rate of disabled individuals.

The premium is paid during the first five years, provided the policyholder is alive and active. The number of years active during the first 5 years of the policy is 4.4714 years. In this example we assume a constant and actuarially fair premium. The premium is calculated using the equivalence principle $E[\mathrm{B}(40,50)] = E[\Pi(40,50)]$ where $E[\Pi(40,50)] = p\ _{40}\overline{L}\overline{D}_1(40,50)$ with $_{40}\overline{L}\overline{D}_1(40,50)$ the discounted sojourn time in the active state between ages 40 and 50 (discounted at age 40). Hence,

$$p = E[\mathrm{B}(40,50)] / {}_{40}\overline{L}\overline{D}_1(40,50) = \frac{0.0784}{4.4714} = 0.01753$$

The figure is the same as that obtained by Haberman and Pitacco (1999, p. 59).

The prospective reserve at age x is defined as the actuarial value of future benefits less the actuarial value of future premiums, measured at age x (x ≥ 40). The actuarial reserve may be given by the state of the policy at x, i.e. by the state occupied by the policyholder at age x. An individual who obtains an insurance policy at age x is likely to be active but may also be disabled, e.g. mildly disabled. In that case, a distinction is made between an *active reserve* and a *disabled reserve* (Haberman and Pitacco, 1999, p. 95). In this paper, the prospective reserve is the active reserve since the individual is assumed to be active at the time of purchase of the disability insurance policy. The actuarial value at x of future benefits is

$$E[\mathrm{B}(x,50)] = b\sum_{t=x}^{49} \exp[-0.039221(t-x+0.5)]\ _{x}\overline{L}_2(t, t+1)$$

where b = 1. The actuarial value at x of the future stream of premiums is

$$E[\Pi(x,50)] = p\sum_{t=x}^{49}\exp[-0.039221(t-x+0.5)]\,\delta(t)\;{}_{x}\overline{L}_1(t,t+1)$$

where p = 0.01753 and δ(t) is one when the policyholder is required to pay a premium at t and zero otherwise. In our example, δ(t) is one for t = 40, 41, … , 44 and zero when t is 45 or larger. $\overline{L}_1(t,t+1)$ is the number of years spent in the active state, irrespective of the state occupied previously; the policyholder may be in the active state since the policy issue or he may be recovered from a disability. The prospective reserve estimated at age x irrespective of the state occupied at x is the *population-based* prospective reserve. The prospective reserve estimated by state of the policy at age x is the *status-based* prospective reserve. The status-based measure differentiates by the state occupied at x. Policyholders who are disabled at x are expected to spend more time in disability beyond x than policyholders who are active at age x. Hence the prospective reserve at x is considerably larger if the policyholder is disabled at x (disabled reserve) than if he is active (active reserve). The difference can be attributed to the expected sojourn time in each state and is independent of the annuity disability benefit and the premium. Note that we assumed that at policy issue (age 40), individuals are active. If policies would be issued to disabled individuals and if they would be required to pay a different premium, the policy would depend on the status at entry, i.e. the policy would be status-based. The multistate model can easily handle status-based policies. The overall or population-based reserve at x is the sum of (1) the disabled reserve times the probability of being disabled at x and (2) the active reserve times the probability of being active at x. The prospective reserve at age x for a policyholder who is disabled at x is close to the number of years between x and policy term (age 50), since b = 1, the rate of recovery from disability is low (μ_{21} = 0.005) and attrition due to death is low. The distinction between population-based and status-based measures is relatively common in multistate demography (see e.g. Willekens, 1987, pp. 136ff).

The population-based prospective reserve is shown in Figure 3. Premiums are paid between the 40th birthday and the 45th birthday. Hence the reserve increases up to age 45 and declines afterwards. At age 45, the reserve is the actuarial value of disability benefits paid by the insurer between ages 45 and 50 since no premium is paid after age 45. The expected number of years in disability between 45 and 50 is 0.0752 (Table 2). Since the annuity benefit is one and no premium is paid after age 45, the prospective reserve at 45 is equal to the expected sojourn time in disability between 45 and 50, discounted at the given constant rate. Who benefits from the financial reserve, which is at its maximum at 45? The future allocation of the reserve may easily be determined. Most of the reserve is for policyholders who are disabled at 45 and a small part is for individuals who are active at 45 but become disabled at a later age but before age 50. The probability that at age 45 a policyholder is disabled is 1 percent. In the absence of recovery and mortality within 5 years and with a zero interest rate, the necessary reserve is 5. Hence the necessary reserve at 45 for policyholders who are active at that age is approximated by the following equation: $0.0752 = 0.97*z + 0.01*5$. From that equation we obtain the prospective active reserve z: $z = \frac{0.0752 - 0.01*5}{0.97} = 0.0260$

The largest part of the prospective reserve of 0.0752 at age 45 is required for policyholders who are disabled at 45 (0.0500 i.e. 66 percent). About 94 percent of the policyholders remains active up to age 50 and about 2 percent dies while in the active state. The premium contributed between ages 40 and 45 covers the disability benefit for policyholders who experience episodes of disability between policy issue at age 40 and policy term at 50 or death before 50.

Figure 3 about here

The multistate actuarial model may be used to assess the effects of prevention of disability relative to cure or care of persons with disability. The outcome of the assessment depends on the actuarial value of the insured benefit which itself depends on the expected number of years in disability and the annuity disability benefit. The expected years (sojourn time) in disability is the outcome of a complex interaction of the transition rates, which are the incidence of disability, the rate of recovery, and the death rate for active and for disabled individuals. If, for instance as a result of improved medical treatment or support for persons with disability, the relative risk of dying for disabled individuals would decline from 1.2 to 1.1, say, then disabled individuals live longer and the actuarial value of the insured benefit increases. The increase is not linear with increased survival because of the discounting. If, on the other hand, the relative risk of dying remains at the initial level, but preventive measures reduce the incidence of disability and hence postpone the age at onset of disability, the total amount of premiums paid increases and the actuarial value of insured benefits decreases. If the age at policy term is fixed, as in our example, preventive measures increase the actuarial value of premiums and reduce the actuarial value of benefits. If the equivalence principle is applied to determine the premium, *preventive measures should result in lower premiums*.

Recently, Vaupel (2005) discussed a method to assess the impact of lifesaving measures on the life expectancy. His conclusion is that the impact depends on the frailty of the individual whose life is saved. By implication, heterogeneity should be taken into account when assessing the impact of lifesaving measures. The analysis can be extended to insurance and multistate actuarial models. A full extension is beyond the scope of this paper. The extension presented here is illustrative only. What is the effect on the prospective reserve of preventing disability for one individual (individual k, say)? We assume that individual k is aged x and is not different from the other individuals of the same age. In other words, individual k experiences the same incidence of disability as other active individuals. The number of years individual k may expect to be disabled before reaching age y depends on the disability status at age x. If individual k is disabled at x and recovery is not possible, the expected number of years of disability before y is the expected number of years lived between x and y. If individual k is active at x, the expected number of years in disability before y is ${}_kL_{12}(x,y,)$:

$$ {}_kL_{12}(x,y) = \int_x^y {}_kS_2(\xi)\,d\xi \,/\, {}_kS_1(x) $$

During the same period, the expected number of years individual k is active is

$$ {}_kL_{11}(x,y) = \int_x^y {}_kS_1(\xi)\,d\xi \,/\, {}_kS_1(x) $$

Note that ${}_kL_{11}(x,y) + {}_kL_{12}(x,y)$ is not equal to y-x because of mortality. Consider a disability following an accident. If an accident is prevented and, as a result, individual k aged x does not become disabled, individual k has an additional ${}_kL_{11}(x,y)$ years of active life before reaching age y. If the accident would have occurred, k would be disabled for the entire period from x to death or age y, whatever comes first. As a result of the prevention of the accident, the expected duration of disability is ${}_kL_{12}(x,y)$ years. Note the assumptions that k is not different from other individuals and that the prevention of the accident does not change the rate of disability.

The actuarial value of the disability annuity foregone and the savings by the insurer, assuming unit-level annuities, is

$$ E\left[{}_1B(x,y)\right] = b\int_x^y \exp\left[-\delta(\xi - x)\right] S_{22}(x,\xi)\,d\xi \,/\, S_2(x) - b\int_x^y \exp\left[-\delta(\xi - x)\right] S_{12}(x,\xi)\,d\xi \,/\, S_1(x) $$

The first term on the right-hand-side is the expected discounted annuity benefit for an individual who is disabled at age x. The second term is the expected discounted annuity benefit for an individual who is active at x. The savings depend on the relative magnitude of the mortality rate of disabled individuals and the incidence of disability. The measure that prevents k to become disabled at x does not prevent disability of occurring at higher ages. It does not prevent disability forever. Its effect is therefore to postpone disability and the associated insurance benefit payments to higher ages. As a result the actuarial value of the stream of disability benefits declines.

The prevention of disability has also an effect on the premium paid. We assume that the insurance premium is paid during the entire period the individual is active. Before the preventive measure, the individual becomes disabled at age x and no premium is paid after that age. After the preventive measure, the actuarial value of the premium collected by the insurer is

$$ E\left[{}_1\Pi(x,y)\right] = \int_x^y \exp\left[-\delta(\xi - x)\right] p_1(\xi)\, S_1(\xi)\,d\xi $$

The increase in the prospective reserve is $E\left[{}_1\mathrm{B}(x,y)\right] - E\left[{}_1\Pi(x,y)\right]$. Unless the premium is adjusted in view of the equivalence principle, the insurer gains from preventive measures.

To determine the impact on the prospective reserve of preventing individual k from becoming disabled, it was assumed that k is not different from the other policyholders. But individuals differ. Individuals with higher frailty levels are more prone to accidents. The neglect of heterogeneity and the differential effects of preventive measures may result in an overestimation or underestimation of the effect. For a discussion of the effect of heterogeneity on the impact of lifesaving measures on the life expectancy, see Vaupel (2005). A similar analysis can be carried out to assess the impact of measures that do not prevent disability but that reduce mortality among disabled individuals, for instance by reducing the rate of progression from mild disability to severe disability.

5. Conclusion

Many decisions we make in life and many events we experience involve the risk of a financial loss. To limit the consequences, the risk is shared by transferring part of the risk to an insurer and by financially compensating the insurer for taking the risk. Financial security throughout the life course calls for life cycle risk management that involves preventive strategies to reduce the likelihood of unwanted events and insurance policies to limit the losses incurred once an unwanted event occurs. Insurance can be associated with any contingency in life. Financial protection is based on the premise that events are predictable and the financial consequences can be estimated. That requires an ability to determine for every event at least three measures: the likelihood of occurrence, the age at occurrence and the financial loss incurred.

This paper presents an instrument for financial life planning. It is a model hat describes the financial lifepaths of cohorts and individual cohort members in the presence of variety of insurance schemes. The model considers several contingencies in the life course and treats the contingencies from a unified perspective. As a result, the conventional distinction between life insurance and non-life insurance is not needed. Instead, generic concepts are used that encompass most insurance schemes in existence today. The concepts are *event* and *state*. The model consists of two modules. The first is a *biographic* model that approaches the life course as a sequence of events and a sequence of states. Events are transitions between the states. The biographic model describes and projects cohort biographies and individual biographies in terms of states occupied and transitions between a state of origin and a state of destination. The transitions are governed by transition intensities that vary with age and that may depend on characteristics of the transitions and the individuals experiencing the transitions. The biographic model is a multistate probability model, more specifically a continuous-time Markov chain. The second module is an *actuarial* model that associates payments with events or transitions and with state occupancies and determines the actuarial value of a single payment or a series of payments.

The biographic actuarial model pictures the financial life course of an individual. Each member of a population has a characteristic set of transition intensities that underlie the life course. To remain practical, individuals who have important characteristics in common are assumed to have transition intensities that differ only as a result of a random factor. One important characteristic that several individuals have in common is the year or period of birth. Membership of a birth cohort is an important attribute in the study of changes in the positions of individuals that compose a population. In this paper, members of the same birth cohort have biographies that differ only as a result of chance. As individuals age, personal attributes change. The dynamics is described by a system of differential equations that may conveniently be studied using matrix methods. The biographic model is entirely in matrix terms and builds on matrix methods that have been developed by Rogers and others in the field of demography. This paper brings to insurance mathematics significant insights from multistate mathematical demography.

The model is illustrated using an example from disability insurance. Premium is paid during some stages of life (active) while insurance benefits are received during other stages (disabled). Survival probabilities differ by disability status and recovery from disability is possible. The example demonstrates the strengths of the matrix formulation of the model and illustrates the contributions that can be expected from mathematical demography. The model specification allows a straightforward extension to multiple functional states, e.g. by distinguishing disability levels.

The biographic actuarial model is an instrument for financial life planning at the individual level and the cohort level. It can effectively be used to determine the transfers between stages of life and between members of a cohort that are required to secure financial protection throughout the life course. The model can easily be extended to multiple cohorts and used to quantify intergenerational transfers in payment schemes that involve several cohorts such as in the PAYGO pension scheme and the more recent notional defined contribution (NDC) schemes.

Increased longevity and increased individual autonomy in lifestyle and life course call for new financial instruments to provide financial security throughout the life course. The instruments should be sufficiently general to encompass individual accounts and traditional social security systems. A lifetime of financial security calls for a holistic approach to life contingencies and life cycle risk management. Scientific methods need to be developed that assist private individuals, financial institutions and governments in maintaining individual financial security in a world characterized by rapid demographic and social change. This paper aimed at contributing to the development of such methods.

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Table 1
State probabilities by age

	Active	Disabled	Dead
40	100.0	0.0	0.0
41	99.4	0.2	0.4
42	98.7	0.4	0.8
43	98.1	0.6	1.2
44	97.5	0.8	1.7
45	96.9	1.0	2.1
46	96.3	1.2	2.5
47	95.7	1.4	2.9
48	95.1	1.6	3.3
49	94.5	1.8	3.7
50	93.9	2.0	4.1

Table 2
Expected sojourn time (in years) in each state for a 40-year old person, by age

	Active	Disabled
40	0.9969	0.0011
41	0.9906	0.0032
42	0.9844	0.0052
43	0.9782	0.0073
44	0.9721	0.0093
45	0.9660	0.0112
46	0.9600	0.0132
47	0.9540	0.0151
48	0.9481	0.0169
49	0.9422	0.0188
Total	9.6923	0.1011

Figure 1. Three-state model of disability

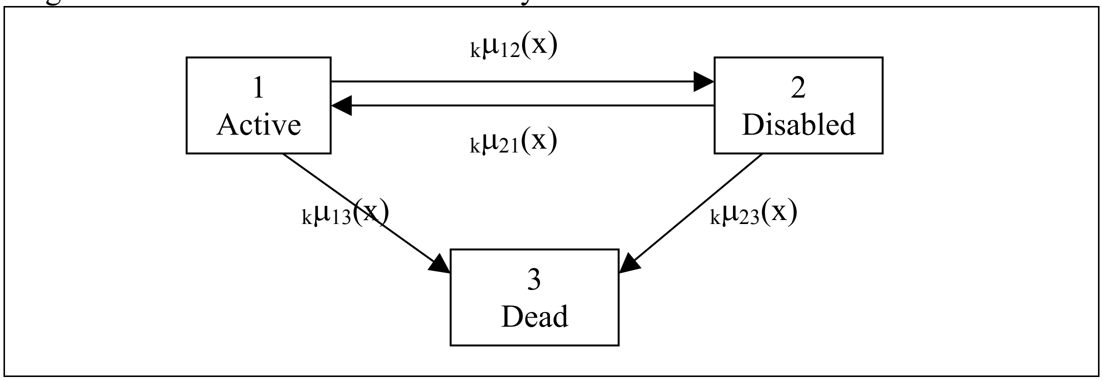

Figure 2
State probabilities by age

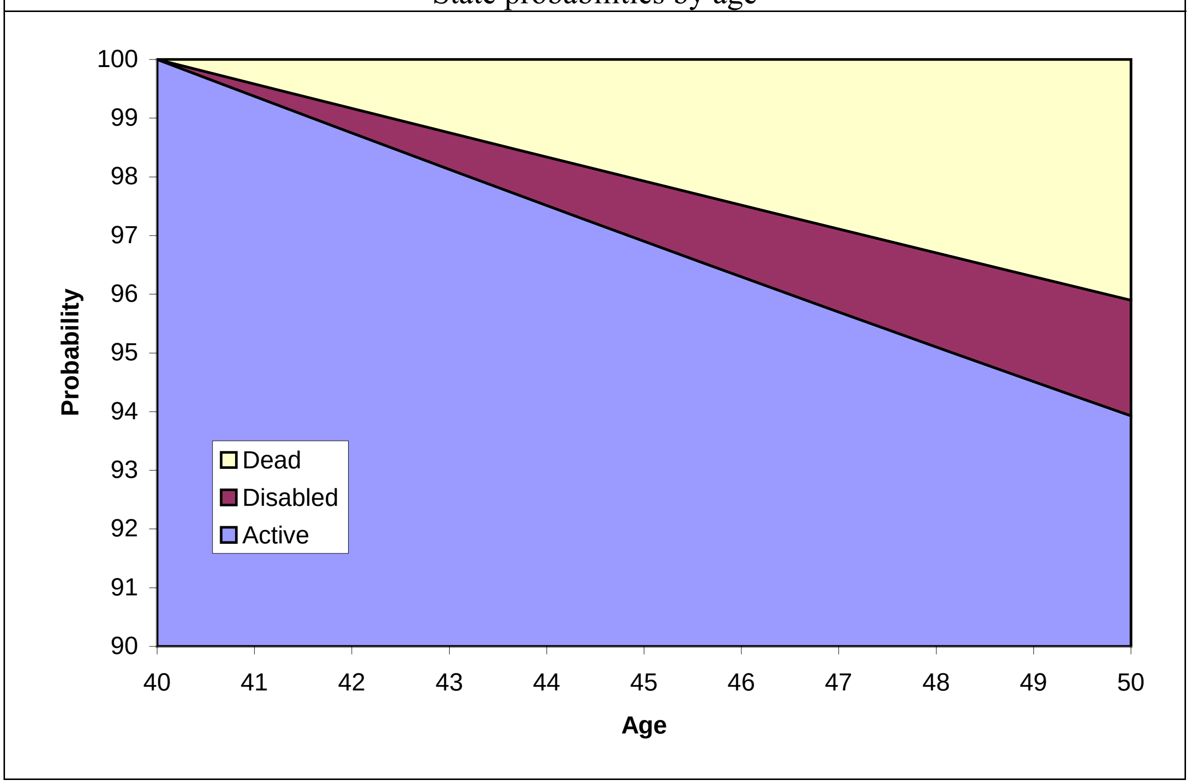

Figure 3
Prospective reserve between policy issue and policy term