Measuring quantum and tempo of vital events by two-dimensional cohort life table functions

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Life table is commonly used for mortality but far less commonly for fertility. Tempo effects have been well known for fertility but yet hardly for mortality. There are obviously discrepancies in measuring and modeling various vital events in demography. However, any vital events can be uniformly modeled as non-repeatable events if they are ordered. And the process of vital events can be represented by life table as the transfer from the inexperienced state to the experienced state by the event. To study the change of life table functions represented by age, they can also be indexed by time. Such life table should be retained as basic or natural, i.e. based on cohort. I call this as two-dimensional cohort life table. This approach is an extension of the definition of mean ages at death by Bongaarts and Feeney (2003).

1.1 cohort life table functions

Let $l(x)$ be the population that has not yet experienced an event and $-\mu(x)$ be the growth rate of the population at age x. Then the following relation is defined. $-\mu(x) = \frac{1}{l(x)}\frac{dl(x)}{dx} = \frac{d\log l(x)}{dx}$ .... (1)

Integrating the both sides from age 0 to age x, it leads to $\left[\log l(x)\right]_0^x = -\int_0^x \mu(a)da$,

Here the population at age 0, $l(0)$ is set as $\mathbf{1}$. Since $l(0)=1$ and $\log l(0)=0$, the following expression is obtained $l(x) = \exp\left[-\int_0^x \mu(a)da\right]$,..(2)

This is the proportion of the population who has not yet experienced the event or more precisely, the proportion of the population at risk of the event. If the intensity of the occurrence $\mu(x)$ is positive at all ages then the equation (2) represents the monotonous decrement process. The change rate is defined as incidence rate as follows, taking its sign positive for the decrease. $d(x) = -\frac{dl(x)}{dx}$ .....…………………………….......(3)

From equation (1), $d(x) = \mu(x)\, l(x)$ ………………………………………………………….(4)

Substituting equation (2) into this, $d(x) = \mu(x)\exp\left[-\int_0^x \mu(a)da\right]$ ....…....………………… (5)

Integrating equation (3) $\int_0^x d(a)da = \left[-l(x)\right]_0^x = 1 - l(x)$, ………………………………………...(6)

Substituting (6) into arranged equation(4), $\mu(x) = \frac{d(x)}{l(x)} = \frac{d(x)}{1-\int_0^x d(a)da}$ ....…………………... (7)

The quantum of the event, Q represents the average frequencies of the event per person, which is identical to the proportion of the population that experiences the event since every person experiences the event only once at most. Integrating the change rate or the incidence rate, $d(x)$ from age 0 to age ω, maximal age of the event, Q is defined as follows. $Q = \int_0^\omega d(x)dx = \int_0^\omega -\frac{dl(x)}{dx}dx = \left[-l(x)\right]_0^\omega = 1 - l(\omega)$, ………………………..(8)

And the tempo index, is defined for those who experience the event as follows,

$$M = \frac{\int_0^{\omega} x d(x) dx}{\int_0^{\omega} d(x) dx} = \frac{\int_0^{\omega} l(x) dx - [xl(x)]_0^{\omega}}{\int_0^{\omega} - \frac{dl(x)}{dx} dx} = \frac{\int_0^{\omega} l(x) dx - \omega l(\omega)}{1 - l(\omega)}, \ldots\ldots\ldots(9)$$

For mortality, $l(\omega)=0$ and Q=1 in equations (8) , which indicates that no person survives ($l(\omega)= 0$) and everyone dies only once (Q=1). Then the life expectancy at birth is given by equation (9) as M= $\int_0^{\omega} l(x) dx$.

1.2 Two-dimensional cohort life table functions

If the dimension of time is added to the equations, the expressions are slightly changed, but no substantial change occurs. However, the change in life table functions is incorporated.

Let $l(x,t)$ be the size at age x and at time t of the population born at time $T = t - x$(constant). Note that time t is the time after x from the time of birth, T = t−x.

Then $l(x,t)$ is expressed as $l(x,t) = l(x, t - x + x)|^{t-x=T}$ and the growth rate of the cohort at time t at age x

$-\mu(x,t)$ is represented by

$$-\mu(x,t) = \frac{1}{l(x,t)} \frac{dl(x, t-x+x)|^{t-x=T}}{dx} = \frac{d \log l(x, t-x+x)|^{t-x=T}}{dx} \ldots\ldots\ldots(10)$$

Integrating the both sides of the equation,

$$[\log l(a, t-x+a)]_0^x = -\int_0^x \mu(a, t-x+a) da \text{ , since } -\mu(x,t) = -\mu(x, t-x+x)|^{t-x=T} .$$

As $l(0, t-x) = 1$ is postulated, similar to equation (2), $l(x,t) = \exp\left[-\int_0^x \mu(a, t-x+a) da\right]$,…..(11)

This equation shows the decreasing process of cohort $l(x,t)$ born at time $t-x$ by the intensity of the event, $\mu(a, t-x+a)$. Then the incidence of the event $d(x,t)$ of cohort born at time t-x is represented by

$$d(x,t)= l(x,t)\, \mu(x,t) \ldots\ldots\ldots(12)$$

Hence, $d(x,t) = \mu(x,t) \exp\left[-\int_0^x \mu(a, t-x+a) da\right]$,…………………(13)

Integrating the boths sides of equation (10) multiplied by $l(x,t)$,

$$\int_0^x d(a, t-x+a) da = -[l(a, t-x+a)]_0^x = 1 - l(x,t) \ldots\ldots\ldots(14)$$

1.3 Tense of cohort life table functions

As equations (11) and (13) indicate, $l(x,t)$ and $d(x,t)$ are composed not only by the intensity at time t $\mu(x,t)$ but also by the intensity at other time before time t. Hence, these can be put in order by the closeness to time t as $\mu(x,t)$, $d(x,t)$ and $l(x,t)$. $l(x,t)$ is most insensitive to the change in the intensity.

1.4 McKendrick equation

$$d(x,t) = -\lim_{a \to 0} \frac{l(x+a, t+a) - l(x,t)}{a} \ldots\ldots\ldots(15)$$

$$= -\lim_{a \to 0} \frac{l(x+a, t+a) - l(x+a, t) + l(x+a, t) - l(x,t)}{a} = -\frac{\partial l(x,t)}{\partial t} - \frac{\partial l(x,t)}{\partial x} \ldots\ldots(16)$$

Or deviding by- $l(x,t)$, the following is obtained. $-\mu(x,t)=\frac{1}{l(x,t)}\frac{\partial l(x,t)}{\partial t}+\frac{1}{l(x,t)}\frac{\partial l(x,t)}{\partial x}$ ....(17)

2.1 Period life table functions and quasi-period life table functions

Period life table functions are defined as follows via $\mu(x,t)$, similarly to equation (2) and (3).

$$l_p(x,t)=\exp\left[-\int_0^x \mu(a,t)da\right],\quad d_p(x,t)=-\frac{\partial l_p(x,t)}{\partial x} \quad \text{……........……........…(18)}$$

Or as $\mu(x,t)=\frac{d_p(x,t)}{l_p(x,t)}$, **similar to equation** (5) , $d_p(x,t)=\mu(x,t)\exp\left[-\int_0^x \mu(a,t)da\right]$……..…(18-2)

Via the incidence rate $d_p(x,t)$, the quantum and the tempo are defined as follows,

$Q3(t)=\int_0^\omega d_p(x,t)dx = \int_0^\omega \mu(x,t)l_p(x,t)dx = 1-l_p(\omega,t)$, ….................…........……………………….(19)

$$M3(t)=\frac{\int_0^\omega xd_p(x,t)dx}{\int_0^\omega d_p(x,t)dx}=\frac{\int_0^\omega l_p(x,t)dx-\omega l_p(\omega,t)}{1-l_p(\omega,t)} \quad \text{………….........………………………………..(20)}$$

For mortality, there is no survivor at maximal age, hence $l_p(\omega,t)=0$. Then, as known well, Q3 (t)= 1 by equation (19), and the life expectancy at birth is M3(t) = $\int_0^\omega l_p(x,t)dx$ by equation (20).

Similarly, via other cohort life table functions, the following quasi-period life table functions are defined.

Quasi-period life table functions (I) via $l(x,t)$: $d_1(x,t)=-\frac{\partial l(x,t)}{\partial x}$, $\mu_1(x,t)=\frac{d_1(x,t)}{l(x,t)}$ ………. (21)

Quasi-period life table functions (II) via $d(x,t)$: $l_2(x,t)=1-\left[\int_0^x d(a,t)da\right]$, $\mu_2(x,t)=\frac{d(x,t)}{l_2(x,t)}$ …..(22)

Via the incidence rates, $d_1(x,t)$ and $d(x,t)$, two sets of quantums and tempos of quasi-period measures can be defined.

$Q1(t)=\int_0^\omega d_1(x,t)dx=\int_0^\omega \mu_1(x,t)l(x,t)dx = \int_0^\omega -\frac{\partial l(x,t)}{\partial x}dx = \left[-l(x,t)\right]_0^\omega = 1-l(\omega,t)$,...............(23)

$$M1(t)=\frac{\int_0^\omega xd_1(x,t)dx}{\int_0^\omega d_1(x,t)dx}=\frac{\int_0^\omega l(x,t)dx-\left[xl(x,t)\right]_0^\omega}{\int_0^\omega -\frac{\partial l(x,t)}{\partial x}dx}=\frac{\int_0^\omega l(x,t)dx-\omega l(\omega,t)}{1-l(\omega,t)}, \quad \text{..........................(24)}$$

$$Q2(t)=\int_0^\omega d(x,t)dx= -\int_0^\omega\left\{\frac{\partial}{\partial t}l(x,t)+\frac{\partial}{\partial x}l(x,t)\right\}dx \quad \text{……………….........……………………(25)}$$

$$=\{1-l(\omega,t)\}\left\{1-\frac{dM1(t)}{dt}\right\}-\frac{dl(\omega,t)}{dt}\{\omega-M1(t)\} \quad \text{…………………………………. (26)}$$

I call this equation as **the total rate equation**, which connects the tempo change to the quantum.

$$M2(t)=\frac{\int_0^\omega xd(x,t)dx}{\int_0^\omega d(x,t)dx} \quad \text{……………………………………………..…………………… (27)}$$

3. Change in the intensity by age

Let $r(x,t)$ be the growth rate of $l(x,t)$, $\frac{1}{l(x,t)}\frac{\partial l(x,t)}{\partial t}$ in equation (17). Then the equation is expressed as

$$r(x,t) - \mu_1(x,t) = -\mu(x,t) \quad \text{......(30)}$$

Substituting $-\mu_1(x,t) = -\mu(x,t) - r(x,t)$ into equation (19-2), then, using equation (18), it leads to

$$l(x,t) = \exp\left[-\int_0^x \{\mu(a,t) + r(x,t)\}da\right] = l_p(x,t)\exp\left[-\int_0^x r(x,t)da\right] \quad \text{......(31)}$$

From this, if $r(x,t) > 0$ for all ages, then naturally, $l(x,t) < l_p(x,t)$.

Attenuating the condition, if $\int_0^x r(x,t)da > 0$, then, $l(x,t) < l_p(x,t)$

Integrating the both sides of equation (31),

$$\int_0^\omega l(x,t)dx = \int_0^\omega l_p(x,t)\exp\left[-\int_0^x r(x,t)da\right]dx \quad \text{......(32)}$$

Then, if $\int_0^x r(x,t)da >$ 0, then, it leads to $\int_0^\omega l(x,t)dx < \int_0^\omega l_p(x,t)dx$ ......(33)

Comparing equations (20) and (24) in this case, the relation between $M1$and $M3$can not be determined.

However, if $l(\omega,t) = l_p(\omega,t)$, then $M1(t) < M3(t)$. For example, for mortality, since $l(\omega,t) = 0$, $l_p(\omega,t) = 0$

and, $M1(t) = \int_0^\omega l(x,t)dx$, $M3(t) = \int_0^\omega l_p(x,t)dx$, then it leads to $M1(t) < M3(t)$ ......(34)

On the contrary, if $\int_0^x r(x,t)da = 0$ then the period life table functions are identical to those of cohort life table and the functions of period to those of quasi-period. In other words, the change in $r(x,t)$ in time causes the discrepancy between the indices of cohort and those of period and between indices of period and those of quasi-period. This model explains the tempo effect in the TFR.

4. Application to first marriage and fertility

The total first marriage rates are derived by the SMAM for Japan calculated by the population census by using total rate equation (equation 26).

Since fertility is specified by the birth order, some contrivance is needed to use the total rate equation. The risk population or the denominator of the hazard rate for parity n increases by n-1-th births and decreases by n-th births (n>1). Hence, the risk population of n-th parity cannot be directly used for the definition of the mean age at birth of n-th order (M1) and the risk population for the calculation of mean age at birth of n-th births is otherwise defined. The decomposition of the TFR in USA is given by using this technique based on the total rate equation.

References

Bongaarts, John and Griffith Feeney, 2003, Estimating mean lifetime, *Proceedings of the National Academy of Sciences of The United States of America*, 100(23): 13127-13133.