On Fisher's Reproductive Value and Lotka's Stable Population

Carlos Galindo

> "The History of Science has suffered greatly from the use by teachers of second-hand material, and the consequent obliteration of the circumstances and the intellectual atmosphere in which the great discoveries of the past were made. A first-hand study is always instructive, and often ... full of surprises."
>
> \- *Ronald Fisher*

Abstract

As Caswell (2001) points out, Reproductive Value (RV) as presented in Fisher 1930 is an abstract concept difficult to interpret. Fortunately, Crow (2002) found the original paper where Fisher (1927) introduced for the first time the idea of a reproductive value. Crow noted one important clarification to Fisher's 1930 formulation: the assumption of ascribing the value of unity to newborns, which is arbitrary, and Fisher did not intend it to be an absolute definition. This is why he also proposed other arbitrary values according to the intended use of RV. Beside this important clarification made by Crow, the original of Fisher (1927) has more explanations that are useful for understanding RV. Here I shall revisit Fisher 1927 and state further important clarifications.

Perhaps even more important than the original of Fisher (1927), there exists one letter from Lotka (1927) in which he explains that all the points made by Fisher were already present in his stable population theory. So Fisher 1927 and Lotka 1927 are long lost documents that can clarify the use of RV and its relationship with stable population theory. Here, guided by Lotka's letter, I shall revisit the original assumptions stated by Lotka and how they were present in Fisher's work. As a corollary, these documents are also helpful to understand the later work made by Keyfitz, and they will also be useful for pointing out several common confusions and misunderstandings about RV and the stable equivalent population.

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Introduction

Caswell (2001, p. 92) cites Fisher's (1930) explanation "that the present value of the future offspring of a person of age x is 'easily seen to be given by the equation'"

$$\frac{v_x}{v_0} = \frac{e^{rx}}{l_x}\int_x^{\infty} e^{-ru} l_u b_u du$$

Then Caswell adds "With all due respect to Fisher, I have yet to meet anyone who finds this equation 'easily seen'" (p. 92). Caswell is right; it is not easy to achieve intuitive understanding of Fisher's Reproductive Value (RV). However, James Crow (2002) recently made a very interesting finding which will help to make Fisher's equation more 'easily seen':

> Fisher (1999) [reprint of 1930] defined reproductive value at age x, v_x, only as a number relative to an arbitrary value of one at age zero, v_0. I, and undoubtedly others, had often wondered why Fisher did not give v_x an absolute meaning, and in fact Keyfitz (1968) later did just that (see also Samuelson 1977). Keyfitz (1968, p. 57) called the reproductive value-weighted total population the 'stable equivalent population'. I had always attributed this to Keyfitz, but in glancing through Fisher's collected papers some years ago, I noticed a throw-away article published in *Eugenics Review* (Fisher 1927). Here, of all places, Fisher gives a definition for the reproductive value at age zero:
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> $$v_0 = 1/\bar{b}\bar{x}_r$$
>
> in which b is the birth rate and x_r is the age of reproduction, both averages being for a population at age stability.
>
> When this value is used, the total reproductive value merges smoothly into the population number as the age structure stabilizes (Crow 1979). It is the same as Keyfitz's stable equivalent population. Why Fisher failed to mention this in his

> 1930 book, written sometime after 1927, is a complete puzzle; it is almost as if Fisher was determined to confuse. In particular, generations of readers of his 1930 book have found the graph on page 28 confusing. This plots the reproductive values of Australian women at about 1911 as ordinate against age. The value at age zero is clearly not unity, as Fisher had led the reader to expect. This discrepancy is finally explained in the variorum edition (Fisher 1999 p. 302) where Fisher, in a letter to C. G. Darwin, dated 16 July 1930, writes: 'I am sorry about v_0. It is unity by definition on page 27 but when I came to make the graph I introduced a factor so as to make the total number of heads in the population in its steady state equal to the total value of such a population. That made v_0 a trifle over 2,' as is apparent from the graph. No wonder the graph is confusing. Incidentally when he invented the idea of the stable equivalent population, Keyfitz was not aware of Fisher's discovery, long before in 1927. Keyfitz was not alone. (Crow 2002, p. 1315)

The goal of Crow's paper is to clarify Fisher's conjecture about natural selection from the "usual elegant obscurity" (p.1314) of Fisher's writings (this conjecture was named the 'Fundamental Theorem of Natural Selection' by Fisher himself). After writing the paragraphs quoted above, Crow moves away from reproductive value and follows the development of Fisher's conjecture. However there are still some other important things to clarify about Crow's finding (for example, the very important point that it is not Keyfitz's, but Lotka's stable equivalent population) and about Fisher's original formulation of RV. Revising Fisher's (1927) original RV formulation will be instructive for three reasons: reproductive value is still an abstract concept and there is no common agreement about its demographic uses and interpretations; the original formulation of RV sheds some light on how Fisher intended this tool to be used; and this paper also gives some clues about the relation of RV with Lotka's stable theory. I will go through Fisher's original formulation in order to show these points.

Fisher's 'original' formulation of RV

It is still commonly believed that Fisher (1930) originally proposed the concept of reproductive value in his book *The genetical theory of natural selection* as a tool for assessing the genetic contribution of one individual to a population, and that afterwards this concept was adapted for comparisons among different populations. However, the concept of RV was originally developed with the purpose of comparing 'potential' reproduction among human populations. In 1927 Fisher wrote a paper for *Eugenics Review* called "The Actuarial Treatment of Official Birth Records", in this paper he proposed for the first time the idea of a reproductive value associated to a certain population. His aim was to develop a technique for establishing the optimal reproduction of the 'working class' given different occupational groups. The calculations were based on men and their sons because Fisher assumed that sons had to remain in the occupational group of their fathers and that women should not enter any occupational group, but he also pointed that calculations should be made on women and their daughters for regional comparisons of populations (as the rest of the eugenists, Fisher held false assumptions about inheritance; in this paper Fisher develops the concept of reproductive value based on mistaken beliefs about the inheritance of mental and working capabilities). Although RV was formulated according to mistaken assumptions, this tool has proven to be useful in Biology and Demography. If we are able to separate the RV statistical tool from the faulty assumptions of Fisher, then his statistical definitions related to different comparisons make sense. As Crow (2002) points out and as Fisher states, the definition for the reproductive value at age zero v_0 was used in order to make female-dominant comparisons of populations.

After the Second World War the members of the Eugenics Society realized the ethical and logical faults of their ideals, or they simply hid their eugenic publications being afraid of social and intellectual discredit. Alas, the original formulation of Fisher was forgotten. However, the ideas that fuelled Fisher's formulation were also present in Lotka's stable theory, which leads to the same definition of v_0 for the population comparisons later intended by Fisher. After reading Fisher 1927, Lotka (1927) sent a letter to *Eugenics Review* where he shows that he had already written the main ideas of Fisher's publication and that several paragraphs can be seen as copies of Lotka's previous work (apparently, given Fisher's reputation at the time, this was not a surprise). Lotka states "That the actuarial principle discussed by Dr. Fisher may be used as an instrument to measure effective fertility is fully set forth, with numerical application, in my paper: *The Measure of Net Fertility, Journal of the Washington Academy of Science*. December, 1925, page 469. A particular detailed application, with much numerical elaboration, of the principle under discussion, has been given by Dublin and Lotka in the paper referred to above, *The True Rate of Natural Increase*. Among other things brought out and illustrated by actual numerical computations in this paper is the separate application of the principle to the female and the male population, a point to which Dr. Fisher also refers". To this letter Fisher merely replied (also in Lotka 1927) that he was not aware of Lotka's work and that "Evidently the only absolutely novel suggestion in my article lies in the estimation of a definite 'reproduction value' for each age of life". The 'novel' suggestion that Fisher claims about a value for each age of life applied to vital records was made, for the first time, by the statistician and epidemiologist William Farr (who worked almost 40 years in the General Register Office of England). William Farr first stated his concept of 'value' for each age of life when looking at economical contributions and Vital Statistics (Farr wrote a book called *Vital Statistics* which is out of print, but the traces of his work can be followed in Humphreys 1885). Later Lotka explained Farr's work, and it will be useful to briefly look at his explanation because it shows the 'economical' or 'financial' rationale behind reproductive value. On the other hand Lotka's work is entirely based in Thermodynamics theory (read Lotka 1998, First part, Principles; in Lotka 1969 p. 9-40) and briefly revising his work will also be useful to clarify the population comparisons intended by Fisher (1927). So, from its very origin, RV is a 'multidisciplinary' tool, and trying to understand the meaning of the concepts used in its development will be helpful for understanding the tool itself.

I will start by following Fisher's (1927) formulation of RV. First he explains that the 'expectation of offspring' of a newly born child can be estimated by "the average number of live births already conceived at each age" by already living persons (p. 104). Today we refer at this indicator as Net Reproduction Rate (considering only one sex for calculating the birth rate).

$$R_0 = \int_0^\infty l_x b_x dx \qquad (1)$$

Fisher then claims that he derives his idea of RV from Malthus' analogy of population increase with compound interest: in this metaphor the date of repayment of a debt is important for the calculation of the rate of interest, and in order to find the appropriate rate in the analogous problem of human increase we must reduce the future children, whose advent we are expecting, to their *present value* (of new born child) equal to unity. This provides an exact measure of the exponential rate of increase of the population (geometrical in discrete case). This means that, if we accept the assumption of what Fisher calls 'the conventional valuation' of newborns as unity v_0=1, then we obtain Lotka's equation. Fisher also points out that Malthus scarcely considered that a decreasing population must be represented by a negative rate of interest, which will occur when the statistical mean of same sex offspring is less than unity. Fisher never mentions it, but the resulting expression is known as the Euler-Lotka equation, in honor of the mathematician that developed it and the ecologist who gave it a biological and demographical use.

$$\int_0^{\infty} e^{-rx} l_x b_x dx = 1 \tag{2}$$

The next step is to add this 'value' to an age group. Fisher claims that the *present value* of the future progeny of an age group, as defined in financial mathematics with the rate of compound interest indicated above, will provide an exact measure of the present value of individuals in the group considered for procreating future generations. He also says that the present value of each individual will increase with age as he (or she) escapes the dangers of infant and child mortality, it will reach its maximum at about 20, and will thereafter decrease as the time for procreation passes, whether such procreation has been realized or not. Then, when reproduction has ceased the reproductive value as a potential ancestor is obviously zero (in his later work on genetics Fisher 1930 referred to this concept as age-specific reproductive value).

$$v_x = \frac{1}{e^{-rx} l_x} \int_x^{\infty} e^{-ru} l_u b_u du \tag{3}$$

The first important clarification to RV is made by Fisher (1927) in his original paper: "The convention that unit value is to be ascribed to the newly born is open to no objection so long as we merely wish to compare the values of different age groups; on the other hand it is not suitable for the comparison of different populations". The term *convention* or *conventional valuation* is a financial one, and it can be translated into plain English as 'you can use any number you want'. This means that some numbers might be more useful than others but there is no 'correct' number for ascribing the value of newborns. This very important point was missed by Crow (2002): there is no 'absolute' definition for the value of the newborns v_0, and this lack of absolute definition gives flexibility to the use of RV. This is what Fisher meant with the term *conventional valuation* and it is important to note it because it is still a confusing point. RV can be used differently according to different assumptions, or conventional valuations, about the number ascribed to the first age group v_0; but we should also be mindful that the results given by RV are highly dependent on the assumptions made about v_0 values and arbitrary population divisions.

Fisher (1927) refers briefly to the ideas of *conventional valuations*, *compounded interest* and *present value*; which indicate the original ideas behind the formulation of reproductive value. But in the original work of Farr this economical thinking is clearly stated. Lotka (1944) explains the original work of Farr (he cites Farr, W. *Journal Stat. Soc. London* 1853, but he gives no title of Farr's publication). Farr was initially trying to estimate the 'capitalized value of human earning capacity'. He equates the *capital value* of a wage-earner to the 'present worth' of his net future earnings. Let the *capital value* be denoted by v_x for a wage-earner of age x and so, he stated

$$v_x = \frac{l_0}{(1+i)^{-x} l_x} \sum_{u=x}^{\omega} (1+i)^{-u+1/2} L_u W_u \tag{4}$$

Where l_x and L_x have the usual demographic meaning, W_x denotes the average of the earners' net annual earnings at age x to x+1, and i denotes the interest rate applied annually to the earnings. Lotka explains that "Since l_0 is a purely arbitrary constant (the *radix* of the life table), we can arbitrarily put l_0=1. Then l_x and L_x, instead of numbers of individuals, represent corresponding proportions. It will simplify our formulae to adopt this convention" (Lotka 1944, p. 10). And here we see again that the assigned value for the first age group (even if here is the radix) is only a *convention*, an *arbitrary assumption*, or in plain English, is just a matter of taste. Even if Farr's calculation is based on discrete time, it is easy to see that if we use birth rates instead of net earnings we obtain an 'annualized' reproductive value. Lotka notes that when the unit of study is the population (not the individuals) it is better to use an instantaneous rate of interest because "the population brings a continuous income, unlike a loan of money, which brings an income at finite intervals" (p. 12). Let r be the interest rate compounded continuously, then the terms that adjust the age-population values $(1+i)^{-x}$ are replaced by e^{-rx}, the approximation made by L_x is no longer necessary, and also the approximation made with the net annual earnings at age x to x+1, denoted by W_x, is no longer necessary (the average annual earnings at age x is denoted by Lotka as w_x). Thus Farr's discrete equation 4 can be expressed in the continuous setting as

$$v_x = \frac{1}{e^{-rx} l_x} \int_x^{\infty} e^{-ru} l_u w_u du \tag{5}$$

So the financial thinking behind reproductive value is as follows: the net reproductive rate, equation 1, can be regarded as a sum of money related to each age (earnings by age in Lotka's explanation or payments by age in Fisher's explanation); in order to compare the different amounts obtained at each age we need to 'move' them towards the same reference time point (age zero), this is done by adjusting the amounts using the prevailing interest rate (the intrinsic growth rate of the population); but the amount related to age zero is going to be our reference amount, so we assign an arbitrary value (a conventional valuation) to this first amount, which yields equation 2; finally, in order to make a fair comparison we need to standardize for the size of the population at each age, which is done dividing by l_x, but this size also has to be moved to the same reference time point $e^{-rx}l_x$. And so we obtain equations 3 and 5 (equation 5 is called the *present value* at age x of a continuous earning or payment stream of w_u beyond age x).

There are two important things to note from Lotka's explanations. The first important thing to note is that when using RV the unit of study is the population, so we cannot ascribe a reproductive value to single individuals. Since RV is an average it would be a mistake to state that a woman or a migrant has certain RV, we can only say that a population (or subpopulation made by an arbitrary division by age or other category) has defined values of RV. Reproductive value as defined in equation 3 is a tool designed for comparisons of reproductive behavior between different age groups: first it 'moves' age-specific values to the same reference point;

then it standardizes these values by the size of the age groups and; finally it compares the standardized values against the arbitrary reference of the first age group. This last point may seem repetitive but is important to emphasize it because, as Crow (2002) mentions, from reading the book of Fisher (1930) many researchers believe that the reproductive value of age zero is 'defined' as the unity. This problem arose mainly from the metaphor used by Fisher of an acquired debt at birth of one life and its subsequent payment by age-contributions (which wrongly leads us to believe that he defined v_0=1 because of this 'one' life that we are supposedly granted), but the important term to note in the metaphor is the *convention* of the debt's value (and as we have seen from Lotka's explanation, the reproductive payments can also be seen as reproductive earnings, and instead of a debt we merely have the present value of such earnings). So once again, Fisher (1927) did not define v_0 as the unity, he explained that v_0 can be any number we want it to be (that is what *conventional valuation* or *arbitrary assumption* mean, that there is no absolute definition for v_0), and he proposed to use v_0=1 when we are interested in comparing RV of different ages within the same population (because it is easy to compare values against the unity but, strictly speaking, we can also use v_0=1000 or v_0=3.1416 and we would still be following Fisher's original formulation of RV). Later I will show that, as Crow (2002) also mentions, Fisher (1927) proposed other conventional valuations or arbitrary values for v_0 intended for other kind of comparisons. That is why the original presentation of the formula of RV made by Fisher (1927) includes the term v_0, and he just notes that v_0=1 if we want to assume this value for a newly born child.

$$v_x = \frac{v_0}{e^{-rx} l_x} \int_x^{\infty} e^{-ru} l_u b_u du \tag{6}$$

I will return to Fisher because he makes the next important clarification, when knowing the values to assign to each age we may evaluate the whole census population RV when the age distribution is also known. He adds that “The comparison of the *total values* of two census populations, unlike the comparison of the mere numbers, provides, when allowance has been made for migration, a simple measure of the population growth or decrease, which may be shown to coincide with the Malthusian rate of interest discussed above, or rather if it is changing, with its value averaged over the intercensal period” (Fisher 1927, p. 106, italics on the original). He also notes that this ratio can be used to test whether the increase in the number of heads of the population is or is not sufficient to counterbalance the increasing average age (in an aging population). What Fisher calls the *total value* is given by

$$V = \int_0^\infty v_x n_x dx = \int_0^\infty \left(\frac{n_x v_0}{e^{-rx} l_x} \int_x^\infty e^{-ru} l_u b_u du \right) dx \qquad (7)$$

Some researchers refer to this concept as net reproductive value, total RV or population RV, but there is also some confusion about it because, once again, this explanation was not made in Fisher 1930. Stearns (1976) explains that Leslie defined the RV for the whole population as the sum of the age specific RV. In the continuous setting the total RV proposed by Leslie is

$$V^* = \int_0^\infty \frac{v_x}{v_0} dx = \int_0^\infty \left(\frac{1}{e^{-rx} l_x} \int_x^\infty e^{-ru} l_u b_u du \right) dx \qquad (8)$$

Clearly the measure proposed by Leslie V^* is different from the one proposed by Fisher V (even though both measures share the remarkable property that their changes in time can be expressed as rV_t or rV_t^*). Leslie’s V^* is merely the sum of the age-specific RV standardized by v_0, while Fisher’s V is not standardized and it also takes into account population size and its age distribution. So Leslie’s V^* can be seen as the Crude or Gross RV of the population, and Fisher’s V can be seen as the Net RV of the population. And once again, the choice of using V^* or V depends on which kind of comparison we want to make.

Last but not least, Fisher (1927) makes a very important explanation about the use of RV (so important that it was the only one reported in Crow 2002): “The convention that unit value is to be ascribed to the newly born is open to no objection so long as we merely wish to compare the values of different age groups; on the other hand it is not suitable for the comparison of different populations. For this purpose a different convention will be more suitable, namely, that in a population in its steady state the total value ascribed to the population is equal to the total number of heads living” (p. 106). So Fisher explains that when comparing different populations the convention of the unity for newborns is not suitable, instead is better to consider the net RV of the population V in its *steady state*, which would be equal to the number of individuals hypothetically alive. Therefore Fisher states that the arbitrary assumption of v_0=1 is not appropriate for comparing different populations, so he proposes another arbitrary assumption (which is the one emphasized by Crow)

$$v_0 = 1/\overline{b}\overline{x}_r \tag{9}$$

Fisher, obsessed with eugenic ideas, explained the reason for this alternative valuation in terms of recruiting sons for their fathers’ occupations: “If two occupations, for example, were each halving their numbers in each generation the rate of decrease *per annum*, and the corresponding need of recruitment, would be greater in that which had the shortest generation, and the sons born early should count for less than the sons born late” (Fisher 1927, p. 105, italics in the original). Disregard Fisher’s ideas of working vocations, the reason for this new arbitrary valuation is that generation rate (stable birth rate) and generation time (stable mean age at childbearing) must be taken into account when comparing different populations. In other words, newborns from populations with different intrinsic birth rates and generation times should not be represented with the same RV. Fisher also explained that this new valuation based upon the *steady state* allows the newborn child to count for more among long-lived people (aged population) than among the short-lived (young population). He also claimed that the comparison of the Net RV of the census population with the Net RV of the population in the *steady state* provides a simple index showing to what extent the actual populations are or are not of ages favorable to reproduction (to his statement it can be added that the problem of comparing different populations is analogous to comparing the same population over time). Fisher clearly noted that in order to make the *steady state* comparison the Euler-Lotka equation should be equated to an initial value named v_0, and v_0 must be chosen so that, in the *steady state*, the total reproductive value V is equal to the population size (equation 9). This is the reason why Fisher (1927) emphasized several times that the value of v_0 is just a convention and this is the reason why he presented the original formula of RV as formula 6. Given the importance of this new valuation, it is justified to ask the following question: what is this *steady state* of the population?

Lotka's stable population

There is one term that Fisher (1927) uses in the last part of his paper that will be fruitful to analyze because it inevitably leads to Lotka's stable population. The term *steady state* comes from the theory of Thermodynamics. By the time Fisher wrote his paper much confusion remained about the use of thermodynamic states to characterize dynamic processes (perhaps outside Physics this confusion still remains in our days). However by 1927 Lotka had already discussed thermodynamic concepts applied to human populations in several papers, and he had already set forward the terms of 'stable' age distribution and 'stable state' of the population (the best example is his influential paper about the intrinsic growth rate, Lotka 1925). This is a very important correction to what Crow (2002) wrote, it was not Keyfitz who 'invented' the idea of stable population; it was Lotka who applied thermodynamic assumptions and calculated stable populations. Later Keyfitz merely proved some of the relations that exist between Lotka's continuous treatment of stable populations and Leslie and others discrete treatments (Goodman 1967). Perhaps in Keyfitz's books the authorship of Lotka's is not clearly stated, but in Keyfitz's original papers he makes it quite clear that Lotka deserves the credit of developing these ideas. And Lotka worked on stable populations long before Fisher wrote about 'steady' states. Since 1911 Sharpe and Lotka were working in problems related to age distributions, and since 1921 Lotka started relating the age distribution problems with thermodynamic 'moving equilibria'.

I will just sketch briefly the ideas behind *steady states*, for detailed and rigorous treatment please read Lotka (1939), or its recent English translation (Lotka 1998), also Zotin (1990) and Haynie (2001). According to the theory of thermodynamics over chemical reactions, systems can be classified in isolated, closed and open systems. Isolated systems exchange nothing with their surroundings (environment), while closed systems exchange only energy and open systems exchange energy and matter. Isolated systems tend towards equilibrium states, while closed and open systems tend towards non-equilibrium states often referred as steady states. "Non-equilibrium steady states of macroscopic systems, whether close to or far from equilibrium, share many of the features of equilibrium states… In fact, the only operational difference between an equilibrium state and a nonequilibrium steady state is that a flux of mass, momentum, or energy is being transported by the system" (Keizer 1984). This is the reason why Lotka (1921) refers to the steady states as *moving equilibria*.

Starting from the assumption of closed systems Lotka (1939) discusses two important steady states for the population: if we assume that the population is a closed system (he names this assumption as *closed population*) then we expect the population to move towards two possible steady states (and the trivial 0 state), the stationary population (originally implied by the life table) and the stable population. These two moving equilibria result from the evolution of the closed system, but they are easier to understand through variations of the steady state assumption;

according to Haynie (2001) in 1925 Haldane and Briggs put forth this assumption: in the steady state the rate of formation of the complex is equal to the rate of its decomposition, so the rate of formation is constant and its velocity is equal to zero. Therefore the stationary population is reached when the rate of formation of the complex (the birth rate) is equal to the rate of its decomposition (the death rate), so the rate of the reaction implied by these two rates (the population growth rate) is constant and equal to zero. The stable population is reached by relaxing the conditions of the stationary population, in this case the rates of formation and decomposition are held also constant but they are not equal, so the rate of reaction (the population growth rate) is constant but different from zero. In his original papers Lotka had no space to explain the thermodynamic basis of his stable population (for example his influential paper of 1925), but in his book (Lotka 1939) he fully explains his assumptions and his sources of inspiration. That non-equilibrium thermodynamics is the basis of the stable population theory is not a well known fact for two reasons: the book of Lotka (1939) was published in French and not until recently (1998) was it translated to English; furthermore, Lotka's book was originally published in two separate volumes, the first one dealt with thermodynamic assumptions and principles explanations, and the second one dealt with their application to human populations. So most of the demographers interested in the topic only acquired the second volume. Nowadays, fortunately, the Spanish (1969) and English (1998) translations include both volumes in the same publication.

There is one main problem with Lotka's assumptions; it will be fruitful to revise it for understanding stable population theory and the use of RV for comparing populations. Lotka is taking literally the closed system assumption for chemical reactions (exchange of energy but not of matter with the environment), so he says that the *closed population* assumption is a population closed to migration (he thought that migration was 'matter' exchange). However, even if population dynamics are somewhat dependent on chemical reactions, the population dynamics do not behave exactly as chemical processes; so the literal translation of chemical closed systems is not suitable for populations. The economic translation for the closed system assumption is far more appropriate (classic economic theory also has its basis in thermodynamics because some of its founding fathers were also physicists, for the complete story read Mirowski and Goodwin 1991, and for some of the mathematic relationships read Smith and Foley 2002); a closed system is translated into economic theory as 'with all the things being the same' or 'all other things being equal'. So the closed population is a *ceteris paribus* population (the environment remains constant, which includes constant natural, social, economic and cultural context). This formulation of the closed population assumption (*ceteris paribus* population) is the correct one, and this fact is easy to see when we realize that the steady assumption can include migration. In a closed population the rate of formation of the complex (birth and immigration rates) and the rate of decomposition (death and emigration rates) are equal, so the rate of formation (population growth rate) also remains constant and its velocity is equal to zero (and so the stationary age distribution is also reached, for the stable age distribution is only needed that birth, death and migration rates remain constant). Several researchers have discussed the fact that the 'no-migration' assumption of Lotka is not necessary for reaching stability, it is only necessary that migration rates remain constant (perhaps the best examples of this result are found in Feeney 1970 and Keyfitz 1971, a longer list of researchers can be found in Cerone 1987). What is called in demography as the closed population assumption allows considering migration (which only affects the absorbing nature of the zero state), and more appropriately biologists refer to this assumption as a population living in a constant environment: "As is clear by now, equilibrium models presume a constant environment" (Reice 2001, p. 17) (do not confound demographic close population and biological constant environment with genetic assumption of close population). The population comparisons as intended by Lotka and Fisher, using the Net RV of the population in its steady state, can include migration rates, although calculations become harder. Furthermore, the tool of RV can be extended to take into account migration rates. Therefore, the demographic assumption of ceteris paribus population is a constant environment population assumption, and the steady states to which this population tends towards can be stationary or stable states.

Not having a well defined closed population assumption has led to confusions in demography, and some of them have not been clarified. To avoid these confusions in subsequent uses of RV and stable population theory is important to emphasize this point: it is obvious that the ceteris paribus population assumption is an astringent simplification of reality; populations are open systems because the environment (everything else) is changing and so demographic rates will not remain constant for the period of time needed to achieve a steady state. Populations are more likely to be found in transient states (which are moving towards steady states but the steady state is almost never reached). Even Fisher (1927) realized that the steady state is only an hypothetical one "Actual populations are seldom at or near the steady state appropriate to their birth and death rates, and could scarcely become so unless the frequency at each age of death and reproduction remained constant for nearly a century" (p. 106).

Therefore the stable equivalent population is only a hypothetically equivalent population and its age distribution and growth rate are not 'true' or 'ultimate' characteristics, they are only hypothetic ones. This may seem quite obvious but several influential papers have confused demographers over quite long time, and it is important to revisit these confusions in order to prevent further misinterpretations of RV. For example, Lotka (1925) got carried away when he named the stable growth rate as the 'true' population growth rate, Goodman (1967) also got carried away when he named it as 'ultimate' or 'eventual' population growth rate. Another example is given by Keyfitz and Flieger (1968) when they define stable population as "Crude rates of birth, death, and natural increase per thousand population, as they would stand ultimately if the observed age-specific rates applied;" (p. 45) to this definition it should be added 'with no changes in their values for a very long period of time'. The stable growth rate is nothing more than the hypothetical rate that follows from the astringent assumption of constant rates. This stable growth rate is a useful decomposition of the actual growth rate (it is 'free' from the effects of the actual age distribution) but if there is any *true* rate of population growth it is the actual or transient growth rate. That is why some researchers are now being more critical with assuming stable characteristics for the populations. For example Koons et al (2005) explain that asymptotic demographic analysis has had a long history of use in population studies, however the stable population state should not be assumed unless empirically justified; and they suggest taking special consideration of the actual population growth rate, which they call *transient growth rate* as opposed to the *stable* or intrinsic growth rate. A corollary of this discussion is that the population comparisons using the Net RV of the stable population, as intended by Lotka and Fisher, are not comparisons of the 'true' or 'eventual' behavior of the population, they are only comparisons of hypothetic populations that would emerge from constant formation and decomposition rates. Keyfitz and Flieger also provide a good example of misinterpretation of RV and stability: "*V* Reproductive value: the expected future girl children who will be born to the existing female population on the observed regime of mortality and fertility, discounted at the intrinsic rate of natural increase" (p. 22). The net reproductive value *V* is not the *expected future* children who *will* be born; it is the *hypothetic* children who *would* be born if the observed vital rates were held constant for long periods of time (and is obvious that the rates will not remain constant for long periods of time). So the comparisons of populations in their steady states using RV, as intended by Lotka and Fisher and done by Keyfitz and Flieger, are not comparisons of ‘future’ or ‘eventual’ states of the populations, they are just comparisons of *hypothetic* populations that would emerge from the *unrealistic* assumption of a *ceteris paribus* population. An important corollary of this discussion is that assumptions are of great importance in population analysis, especially when it comes to interpreting results.

It is also important to revisit another source of confusion. What Fisher (1927) wrote is that, if we want to compare the reproductive value of different populations (or the same population in different time points) it is not suitable to use the arbitrary assumption of v_0=1. Instead is better to use another arbitrary assumption for v_0, namely the one that allows V, the Net RV, of the stable population to be equal to the size of this hypothetical population (such arbitrary assumption is given by equation 9). When Lotka read Fisher’s (1927) paper he realized that he had already proposed this idea along with his stable population theory, later (Lotka 1928, 1939 and 1942) he formalized the fertility relationships in stable populations. Within these relationships Lotka introduced the *constants* Q, which are the roots of the so called ‘renewal equation’. Only one of these roots is real and is associated with the stable growth rate. In his papers Lotka used several different notations for this root, in his book (1939) he named it Q_ρ and he used it to calculate a constant K, which he used to calculate the size of the stable population. Later Keyfitz (1939) named this constant K as ‘stable equivalent value’ but he denoted it by Q. Keyfitz also proved that “In fact, V [with conventional valuation v_0=1] is a simple multiple of Q. In the continuous representation, V is exactly equal to Q multiplied by the intrinsic birth rate b and by the mean age at childbearing” (Keyfitz and Caswell 2005, p. 204). Revisiting the changes in the notation is important because is easy to get confused between the *real root* of the ‘renewal equation’ and the *stable equivalent value*. The stable equivalent value Q = V with the arbitrary assumption of equation 9, but the real root of the 'renewal equation' denoted by Keyfitz as Q_1 is V with the following arbitrary assumption

$$v_0 = 1/\bar{x}_r \tag{10}$$

So it is important to realize that Q and Q_1 are different concepts: Q is referred as the stable equivalent value; and the real root Q_1 as the stable equivalent births. Both Q and Q_1 are V with different conventional valuations for v_0. If Keyfitz was unaware of the original paper of Fisher (1927) and the letter from Lotka (1927), then he rediscovered the content of Lotka's letter: the main ideas of Fisher 1927 had already been posed in Lotka's stable population theory. Furthermore, if Fisher never explained to Keyfitz his original ideas for RV (even though they were acquaintances, Bennett 1990), then Keyfitz rediscovered two of the main points of Fisher (1927): the main utility of RV rests in the non-absolute valuation of v_0, and the conventional valuation given by equation 9 is useful for comparing populations in their *steady state*. This may seem just as a mathematical curiosity, but it has had major influence in demography. Keyfitz (1971) used V with the arbitrary valuations given by equations 10 and 9, which he called stable equivalent births Q_1 and stable equivalent of the total population Q_0 (in his latter books he denoted it only by Q), to develop the concept of *population momentum* for an immediate drop of fertility to replacement level

$$\frac{Q}{N_0} = \frac{b\bar{x}_d}{r\bar{x}_r}\left(\frac{R_0 - 1}{R_0}\right) \tag{11}$$

Where Q is the stable equivalent value, N_0 is the size of the initial population, r and b are the stable growth and birth rates, x denotes age so in the upper part there is the mean age at death (life expectancy) and in the lower part the mean age at reproduction (childbearing), finally R_0 denotes the net reproductive rate. The population momentum is clearly a multiple of the stable equivalent value, so it is also V with a different conventional valuation, namely

$$v_0 = 1/N_0\overline{b}\overline{x}_r \qquad (12)$$

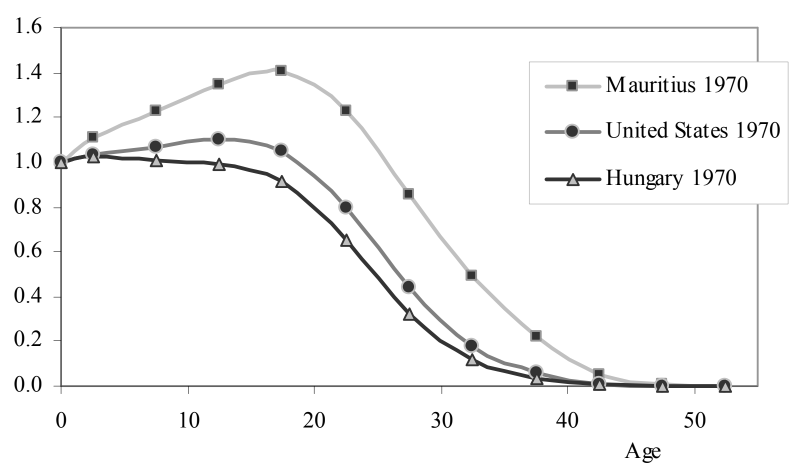
Besides the importance of this concept for mathematical demography, the influential paper about *population momentum* was clearly written to support already existing birth control and family planning programs: “In some countries hesitation in making contraception available is rationalized by the view that the country is not yet ‘full’. Governments as far separated as those of Brazil today and of Indonesia in Sukarno’s time refer to plentiful land and other resources as evidence that their populations could stand considerable further increase before they must become stationary. Concern that total numbers will taper off prematurely is misplaced. If presently high-fertility countries were to experience an immediate drop to age-specific birth rates that would just replace existing parents, the ultimate stationary population would be about two thirds higher than the present total” (Keyfitz 1971, 71). So the rediscovery of the non-absolute valuation for newborns RV has been of importance also in applied demography and in supporting population policies (for the influence of these programs and policies on Demography see Caldwell and Caldwell 1986).

However, confusion still remains. For example, the paper of Keyfitz (1971) presents misleading terms and assumptions like ‘ultimate size’, ‘must become stationary’, and so on (problems obviously derived from misinterpretations of closed and open assumptions leading to steady states of the population). In this sense it is important to notice that the idea (and the formula) of *population momentum* was already stated in Fisher 1927. And Fisher’s interpretation of formula 11 is more appropriate because he was not thinking about ‘eventual’ or ‘ultimate’ states of the population but about comparisons with potential or hypothetic states: “the comparison of the census population with its total value [V calculated using equation 9] provides a simple index showing to what extent the actual populations are or are not of ages favorable to reproduction" (Fisher 1927, p. 106-107).

Discussion

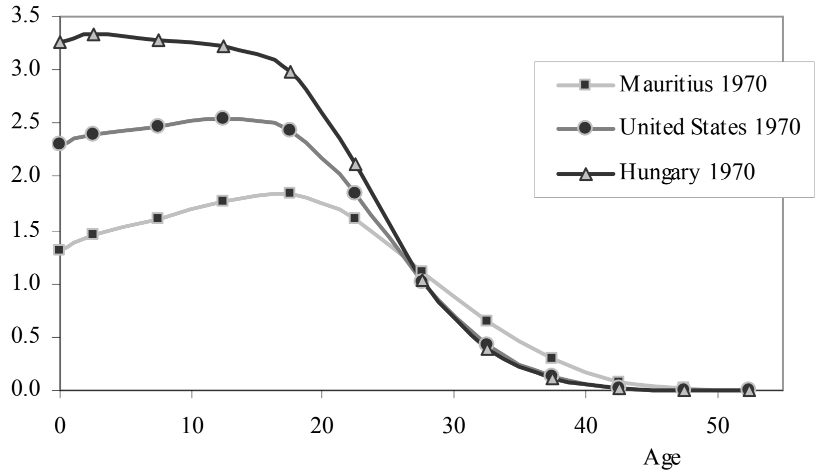
Even though the paper of Crow (2002) has already called for a revision of the 'usual' valuation of RV, there is still confusion about the use of arbitrary assumptions in RV (some are clearly in contradiction with the original paper of Fisher 1927). For example, when explaining reproductive value Keyfitz and Caswell (2005) compare Mauritius, United States and Hungary for years close to 1970 (they use data from Keyfitz and Flieger 1971); but they use the arbitrary assumption v_0=1, which Fisher described as inappropriate for population comparisons. The comparison made in Keyfitz and Caswell 2005, for the same countries, for the year 1970 is shown in Figure 1 (data from Keyfitz and Flieger 1990). On the other hand, Figure 2 shows the same comparison with Fisher's suggestion of using the arbitrary assumption of equation 9. Fisher's (1927) reasons for this suggestion can be summed up in the following quote: "The convention of valuation based upon the steady state allows the new born child to count for more among long lived people [aging population] than among the short lived [young population], as he obviously ought to do" (p. 106).

Figure 1, Misleading comparison of Reproductive Values



Source: Own elaboration, data from Keyfitz and Flieger (1990)

Figure 2, Appropriate comparison of RV using equation 9



Source: Own elaboration, data from Keyfitz and Flieger (1990)

Reproductive value is a useful tool. As Keyfitz (1985) explains, it can be used to answer question about the numerical effects of sterilization, mortality and emigration at particular ages; it also allows demographic analysis of eradicating specific diseases. Furthermore, researchers like Michod (1979), Koons et al. (2005) and Ediev (2001) are proposing new uses of RV for understanding evolutionary mechanisms and population history and prospects. But when looking Figures 1 and 2, it becomes obvious that the effect of the comparisons and interpretations of RV are highly dependent on the conventional valuations of v_0. So we should not forget that conclusions taken from RV are highly dependent on the assumptions made during RV calculations. Keeping this in mind becomes even more important when we remember that Fisher was a leading eugenist and that his quantitative conclusions about the reproductive potential of societies (and about humankind in general) were faulty because of his erroneous assumptions on arbitrary population divisions. Therefore, the most important point to clarify about RV is that when calculated on an objective scale, the measure of reproductive value implies absolutely nothing about social values, racial superiorities, etc.

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